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UNIT - 4

LAPLACE TRANSFORM

- F.T is used to find the frequency information of a signal /system
- L.T is a complex F.T & can be used to find the pole-zero response of a system

$\Rightarrow$ $$x(t) \xleftrightarrow{FT} X(\omega)$$

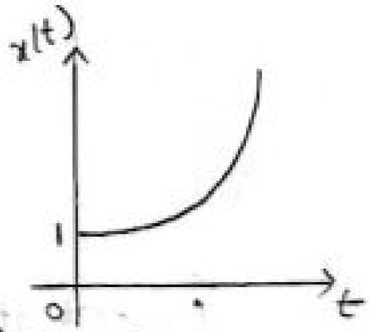
$$X(\omega) = FT[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$x(t) = IFT\,[X(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

- condition for the existance of FT $\Rightarrow \int_{-\infty}^{\infty} |x(t)|\, dt < \infty$

Eg :- $x(t) = e^{2t}\, u(t)$

$$\int_{-\infty}^{\infty} e^{2t}\, u(t) \cdot dt = \int_{0}^{\infty} e^{2t}\, dt = \left[\frac{e^{2t}}{2}\right]_0^{\infty}$$



$$= \frac{1}{2}\left[e^{\infty} - e^{0}\right] = \infty$$

$\therefore$ F.T doesn't exist [since it is not absolutely integrable]

$\Rightarrow$ $$x(t) \xleftrightarrow{LT} X(s)$$

$$X(s) = LT\,[x(t)]$$

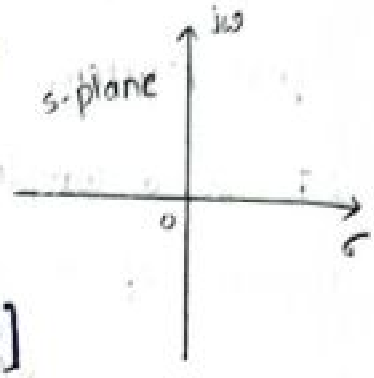
$$x(t) = ILT\,[X(s)]$$

's' is complex freq variable $[s = \sigma + j\omega]$

- $X(s) = LT[x(t)] = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} dt$

$$= \int_{-\infty}^{\infty} x(t) \cdot e^{-(\sigma + j\omega)t} dt$$

$$= \int_{-\infty}^{\infty} x(t) \cdot e^{-\sigma t} \cdot e^{-j\omega t} dt$$



$$LT[x(t)] = FT[e^{-\sigma t} \cdot x(t)]$$

if $\sigma = 0 \Rightarrow LT[x(t)] = FT[x(t)]$

- condition for existance of L.T $\Rightarrow \int_{-\infty}^{\infty} |e^{-\sigma t} \cdot x(t)| dt < \infty$

<u>Eg</u>: $x(t) = e^{2t} u(t)$

$$\int_{-\infty}^{\infty} |e^{-\sigma t} e^{2t} u(t)| dt = \int_{0}^{\infty} e^{-(\sigma - 2)t} dt$$

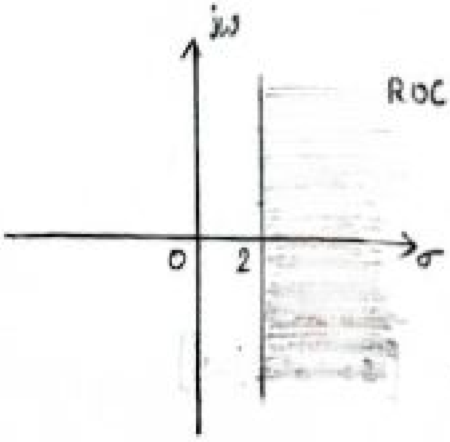
$$= \left[\frac{e^{-(\sigma-2)t}}{-(\sigma-2)} \right]_0^{\infty}$$

$$= -\frac{1}{\sigma - 2} \left[e^{-(\sigma-2)\infty} - e^0 \right]$$

$\int_{0}^{\infty} |e^{-\sigma t} \cdot e^{2t} u(t)| dt < \infty$ if $\sigma - 2 > 0$

$$\sigma > 2$$

$$Re(s) > 2$$



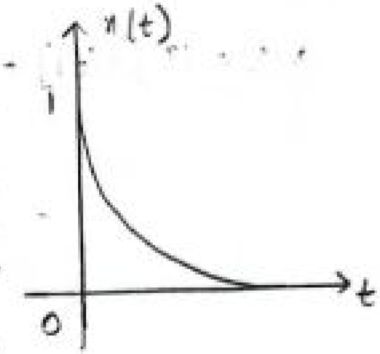
- Region of convergence :- where LT exists

$2 < \sigma < \infty$

$-\infty < \omega < \infty$

Q, Find the L.T. of signal, $x(t) = e^{-at}u(t)$ -

$$X(s) = LT[x(t)] = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} dt$$



$$= \int_{-\infty}^{\infty} e^{-at} u(t) \cdot e^{-st} dt$$

$$= \int_{0}^{\infty} e^{-(s+a)t} \cdot dt$$

$$= \left[\frac{e^{-(a+s)t}}{-(a+s)} \right]_0^{\infty}$$

$$= -\frac{1}{s+a}\left[e^{-(s+a)\infty} - e^{0} \right] \quad - (1)$$

ROC :- $\int_{-\infty}^{\infty} |e^{-\sigma t} \cdot x(t)| dt = \int_{0}^{\infty} e^{-\sigma t} \cdot e^{-at} dt = \left[\frac{e^{-(\sigma+a)t}}{-(\sigma+a)} \right]_0^{\infty}$

$$= -\frac{1}{\sigma+a}\left[e^{-(\sigma+a)\infty} - e^{0} \right]$$

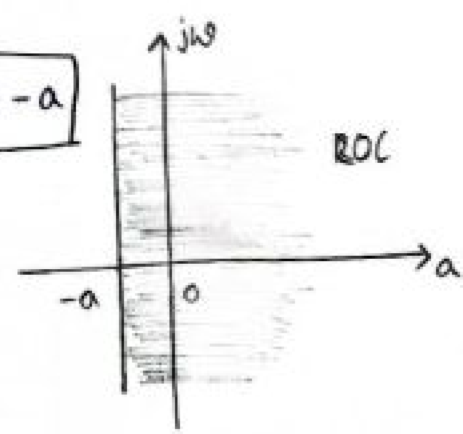
$$\int_{-\infty}^{\infty} |e^{-\sigma t} \cdot x(t)| dt < \infty \quad \text{if} \quad \sigma + a > 0$$

$$\sigma > -a$$

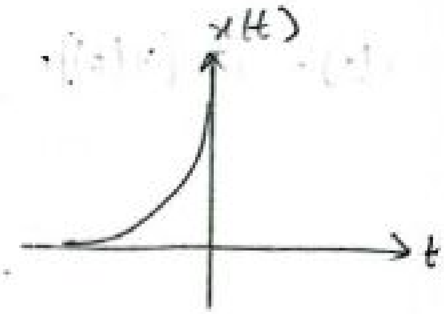
from (1), $X(s) = -\frac{1}{s+a}\left[e^{-(\sigma + j\omega + a)\infty} - 1 \right]$

$$= -\frac{1}{s+a}\left[\underbrace{e^{-(\sigma+a)\infty}}_{0} \cdot e^{-j\omega\infty} - 1 \right] \quad \text{if} \quad \sigma + a > 0$$

$$\boxed{X(s) = \frac{1}{s+a}} \qquad \boxed{\text{ROC :- } \sigma > -a}$$



2Q Find the L.T of signal $x(t) = e^{at}\, u(-t)$; $a > 0$ -



$$X(s) = LT[x(t)] = \int_{-\infty}^{\infty} e^{at}\, u(t) \cdot e^{-st}\, dt$$

$$= \int_{-\infty}^{0} e^{at}\, e^{-st}\, dt$$

$$= \int_{-\infty}^{0} e^{-(s-a)t}\, dt$$

$$X(s) = -\frac{1}{s-a}\left[e^{0} - e^{(s-a)\infty}\right]$$

ROC :- $\int_{-\infty}^{\infty} |e^{-\sigma t} \cdot x(t)|\, dt = \int_{-\infty}^{0} e^{-\sigma t} \cdot e^{at}\, dt$

$$= \int_{-\infty}^{0} e^{-(\sigma - a)t}\, dt$$

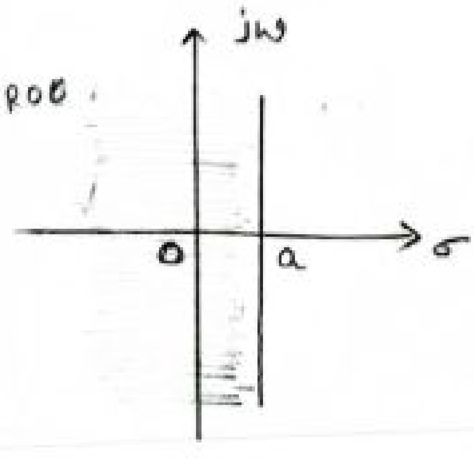
$$= -\frac{1}{\sigma - a}\left[e^{0} - e^{(\sigma - a)\infty}\right]$$

$\int_{-\infty}^{\infty} |e^{-\sigma t} \cdot x(t)|\, dt < \infty$ if $\sigma - a < 0$

$\sigma < a$

✓

$$\boxed{X(s) = \frac{-1}{s-a}} \qquad \boxed{\text{ROC :- } \sigma < a}$$



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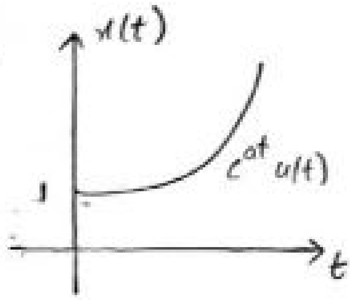
3Q, find the LT of signal $x(t) = e^{at}\, u(t)$; $a > 0$ -

$$X(s) = LT[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$



$$= \int_{-\infty}^{\infty} e^{at}\, u(t)\, e^{-st}\, dt$$

$$= \int_{0}^{\infty} e^{-(s-a)t}\, dt$$

$$= \frac{-1}{(s-a)} \left[e^{-(s-a)\infty} - e^{0} \right]$$

$$= -\frac{1}{s-a} \left[e^{-(\sigma+j\omega-a)\infty} - 1 \right]$$

$e^{+\infty} = \infty$

$e^{-\infty} = 0$

$$= \frac{1}{s-a} \left[1 - \underbrace{e^{-(\sigma-a)\infty}}_{0} \cdot \underbrace{e^{-j\omega\infty}}_{\text{const}} \right]$$

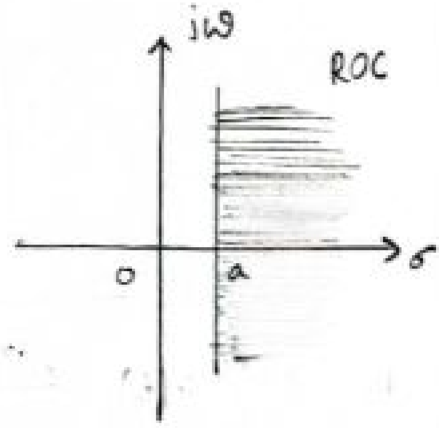
✓

$$\boxed{X(s) = \frac{1}{s-a}}$$

If $\sigma - a > 0$

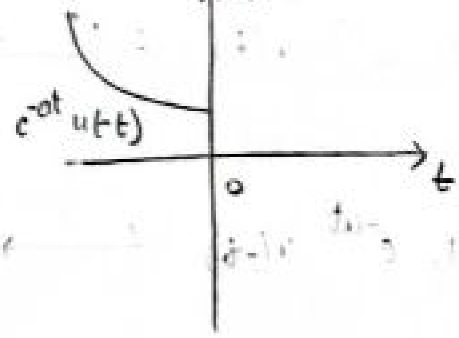
$\sigma > a$

$\boxed{\text{ROC :- } \sigma > a}$



4Q, find the LT of $x(t) = e^{-at}\, u(t)$; $a > 0$

$$X(s) = LT[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$



$$= \int_{-\infty}^{\infty} e^{-at}\, u(-t)\, e^{-st}\, dt$$

$$= \int_{\infty}^{0} e^{-(a+s)t} \, dt$$

$$= \frac{-1}{s+a}\left[e^{-(s+a)\infty} - e^{0}\right]$$

$$= -\frac{1}{s+a}\left[1 - e^{(\sigma+j\omega+a)\infty}\right]$$

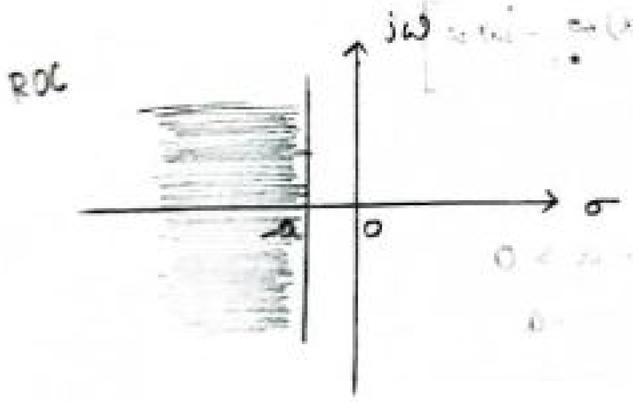
$$= -\frac{1}{s+a}\left[1 - e^{(\sigma+a)\infty} \cdot e^{j\omega\infty}\right]$$

$$\boxed{X(s) = -\frac{1}{s+a}}$$

$\sigma + a < 0$

$\sigma < -a$

$$\boxed{\text{ROC} :- \sigma < -a}$$



1, $e^{-at} u(t) \xleftrightarrow{LT} \frac{1}{s+a}$ ROC : $\sigma > -a$

2, $e^{at} u(-t) \xleftrightarrow{LT} -\frac{1}{s-a}$ ROC : $\sigma < a$

3, $e^{at} u(t) \xleftrightarrow{LT} \frac{1}{s-a}$ ROC : $\sigma > a$

4, $e^{-at} u(-t) \longleftrightarrow \frac{-1}{s+a}$ ROC : $\sigma < -a$

Properties of ROC :-

1, For causal signal, ROC is $\sigma > \sigma_{max}$

2, for anti-causal signal, ROC is $\sigma < \sigma_{min}$

- ROC specification is must to calculate ILT

5, Find the LT of $x(t) = \delta(t)$

$$X(s) = LT[x(t)] = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} dt.$$

$$= \int_{-\infty}^{\infty} \delta(t) \cdot e^{-st} dt$$

Apply sampling property of an impulse

$$\int_{-\infty}^{\infty} y(t) \cdot \delta(t) \cdot dt = y(t) \Big|_{t=0}$$

$$= e^{-st} \Big|_{t=0}$$

$$= e^{0} = 1$$

$\boxed{X(s) = 1}$ $\boxed{\text{ROC : entire s-plane}}$

6, Find the LT of $x(t) = u(t)$

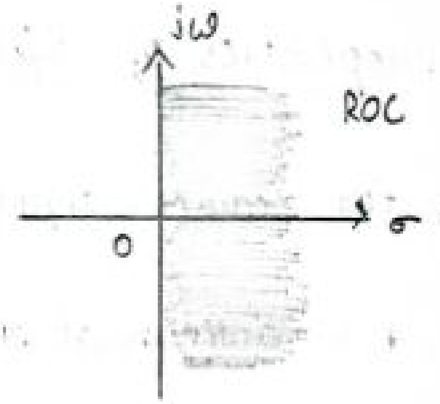
$$X(s) = LT[x(t)] = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} dt$$

WKT

$$y(t) = e^{-at} u(t) \xleftrightarrow{LT} Y(s) = \frac{1}{s+a} \quad \text{ROC} : \sigma > -a$$

Put $a = 0$

$$\boxed{u(t) \xleftrightarrow{LT} \frac{1}{s}} \qquad \boxed{ROC : \sigma > 0}$$



7, Find the LT of $x(t) = \begin{cases} e^{at} & ; 0 \le t \le T \\ 0 & ; \text{else} \end{cases}$

$$X(s) = LT\,[x(t)] = \int_{-\infty}^{\infty} x(t) \cdot e^{-st}\, dt$$

$$= \int_{0}^{T} e^{at} \cdot e^{-st}\, dt$$

$$= \int_{0}^{T} e^{-(s-a)t}\, dt$$

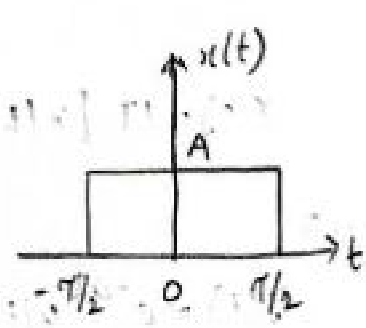
$$= \frac{-1}{s-a} \left[e^{-(s-a)T} - e^{0} \right]$$

$$\boxed{X(s) = \frac{1 - e^{-(s-a)T}}{s-a}} \qquad \boxed{ROC : \text{entire } s\text{-plane}}$$

- for finite duration signals, the ROC is entire s-plane

8, Find the LT of rectangular pulse

$$x(t) = A \cdot \text{rect}\,(t/T)$$



$$X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-st}\, dt$$

$$= \int_{-T/2}^{T/2} A \cdot e^{-st}\, dt$$

$$= \frac{-A}{s}\left[e^{-s \cdot T/2} - e^{sT/2}\right]$$

✓

$$\boxed{X(s) = A\left[\frac{e^{sT/2} - e^{-s \cdot T/2}}{s}\right]}$$

$\boxed{\text{ROC :- entire s-plane}}$

9, Find the ROC of $x(t) = e^{-(1-3j)t}\, u(t)$

ROC :- $\int_{-\infty}^{\infty} |e^{-\sigma t} x(t)|\, dt < \infty$

$$= \int_{-\infty}^{\infty} |e^{-\sigma t} \cdot e^{-(1-3j)t} \cdot u(t)|\, dt$$

$$= \int_{0}^{\infty} |e^{-\sigma t} \cdot e^{-t} \cdot e^{3jt}|\, dt$$

$$= \int_{0}^{\infty} |e^{-(\sigma+1)t}|\, \underbrace{|e^{3jt}|}_{=1} \cdot dt$$

$$= \int_{0}^{\infty} e^{-(\sigma+1)t}\, dt$$

$$= \frac{-1}{\sigma+1}\left[e^{-(\sigma+1)\infty} - e^{0}\right]$$

$$= \frac{1}{\sigma+1}$$

$\sigma + 1 > 0 \Rightarrow \sigma > -1$

$$\boxed{\text{ROC} : \sigma > -1}$$

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10) Find the L.T of signal $x(t) = e^{-j\omega_0 t} u(t)$

W.K.T $\quad e^{-at} u(t) \xleftrightarrow{LT} \frac{1}{s+a} \qquad ROC : \sigma > -a$

$$\boxed{e^{-j\omega_0 t} u(t) \xleftrightarrow{LT} \frac{1}{s+j\omega_0}}$$

$s = -j\omega_0$

Real part $= 0$

$\therefore$ ROC : $\sigma > 0$

11) Find the L.T of $x(t) = e^{j\omega_0 t} u(t)$

$$\boxed{e^{j\omega_0 t} u(t) \xleftrightarrow{LT} \frac{1}{s-j\omega_0}} \qquad ROC : \sigma > 0$$

Properties of laplace Transform

1) linearity property :

If $\quad x_1(t) \xleftrightarrow{LT} X_1(s) \qquad Roc : R_1$

$\quad x_2(t) \xleftrightarrow{LT} X_2(s) \qquad ROC : R_2 \qquad$ then

$$\alpha x_1(t) + \beta x_2(t) \xleftrightarrow{LT} \alpha X_1(s) + \beta X_2(s) \qquad ROC : R_1 \cap R_2$$

Proof :-

$$L[\alpha x_1(t) + \beta x_2(t)] = \int_{-\infty}^{\infty} [\alpha x_1(t) + \beta x_2(t)] e^{-st} \, dt$$

$$= \alpha \int_{-\infty}^{\infty} x_1(t) e^{-st} dt + \beta \int_{-\infty}^{\infty} x_2(t) e^{-st} dt$$

$$= \alpha \, L[x_1(t)] + \beta \, L[x_2(t)]$$

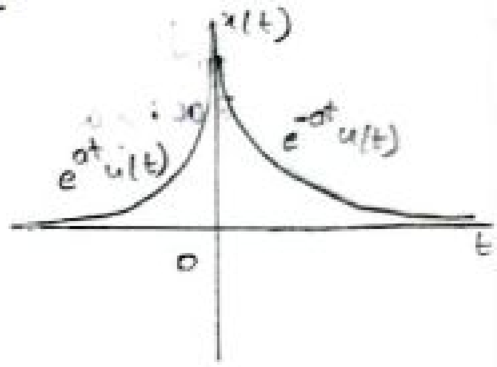
$$\boxed{\alpha x_1(t) + \beta x_2(t) \xleftrightarrow{LT} \alpha X_1(s) + \beta X_2(s)}$$

IMP SM

Eg: find L.T of $x(t) = e^{-a|t|}$ for ① $a>0$ ② $a<0$

(i) $a > 0$:- $x(t) = e^{-a|t|} = \begin{cases} e^{-at} & \text{for } t \geq 0 \\ e^{at} & \text{for } t < 0 \end{cases}$

$$x(t) = e^{-a|t|} = e^{-at} u(t) + e^{at} u(-t)$$

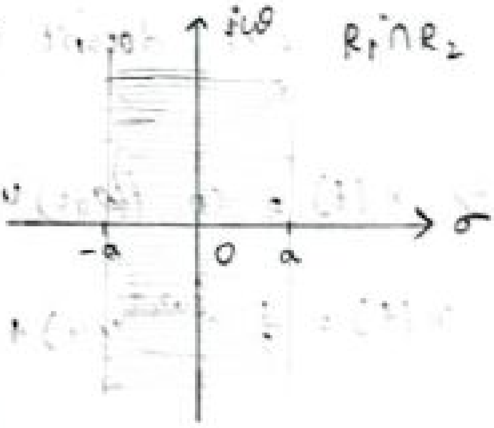


Apply linearity prop

$$L[x(t)] = L[e^{-at} u(t)] + L[e^{at} u(-t)]$$

$$X(s) = \frac{1}{s+a} - \frac{1}{s-a}$$

$\downarrow$ $R_1 : \sigma > -a$ $\downarrow$ $R_2 : \sigma < a$



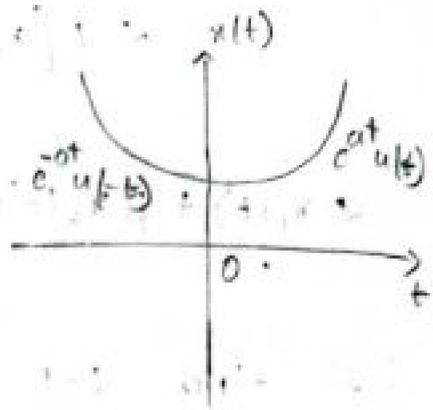
$$\boxed{X(s) = \frac{-2a}{s^2 - a^2} \quad ROC : -a < \sigma < a}$$

Note :

For two-sided signal, ROC lies b/w two intervals i.e

$$\boxed{\sigma_{min} < \sigma < \sigma_{max}}$$

ii, $a < 0$:- $x(t) = e^{-a|t|} = \begin{cases} e^{at} & \text{for } t \geq 0 \\ e^{-at} & \text{for } t < 0 \end{cases}$



$x(t) = e^{-a|t|} = e^{at}u(t) + e^{-at}u(-t)$

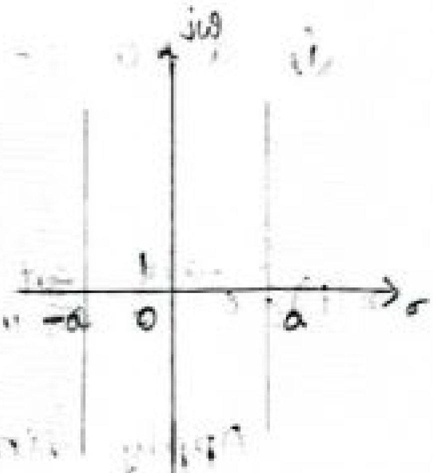
Apply L.P

$L[x(t)] = L[e^{at} \cdot u(t)] + L[e^{-at}u(-t)]$

$X(s) = \frac{1}{s-a} - \frac{1}{s+a}$

$\downarrow$ $\downarrow$

$R_1: \sigma > a$ $R_2 = \sigma < -a$



common ROC = ϕ

$\therefore$ L.T doesn't exist

2) $x(t) = \cos(\omega_0 t)u(t)$

$x(t) = \frac{1}{2}e^{j\omega_0 t}u(t) + \frac{1}{2}e^{-j\omega_0 t}u(t)$

$e^{-j\omega_0 t}u(t) \xleftrightarrow{LT} \left[\frac{1}{s+j\omega_0}\right]$ ROC: $\sigma > 0$

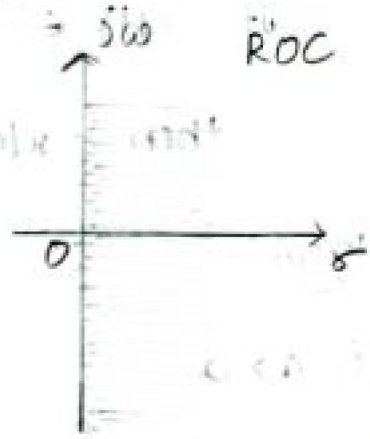
$e^{j\omega_0 t}u(t) \xleftrightarrow{LT} \left[\frac{1}{s-j\omega_0}\right]$ ROC : $\sigma > 0$

Apply L.P

$L[x(t)] = \frac{1}{2}L\left[e^{j\omega_0 t}u(t)\right] + \frac{1}{2}L\left[e^{-j\omega_0 t}u(t)\right]$

$$X(s) = \frac{1}{2}\left[\frac{1}{s-j\omega_0}\right] + \frac{1}{2}\left[\frac{1}{s+j\omega_0}\right] \qquad \text{ROC} : \sigma > 0$$

$$X(s) = \frac{s}{s^2+\omega_0^2}$$



$$\boxed{\cos(\omega_0 t)\, u(t) \xleftrightarrow{LT} \frac{s}{s^2+\omega_0^2}}$$

ROC : $\sigma > 0$

3) $x(t) = \sin(\omega_0 t)\, u(t)$

$$x(t) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$x(t) = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t} u(t)$$

$$\cancel{x(t) =} \quad e^{-j\omega_0 t} u(t) \xleftrightarrow{LT} \frac{1}{s+j\omega_0} \qquad \text{ROC} : \sigma > 0$$

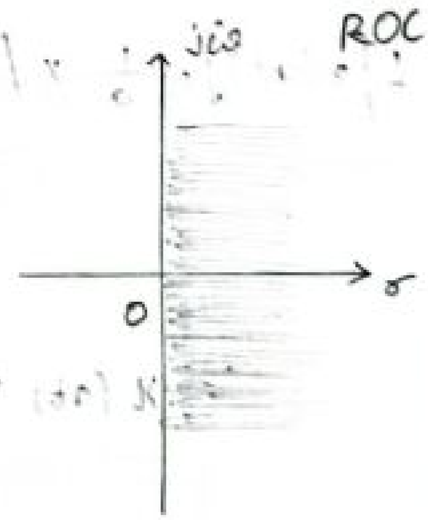
$$e^{j\omega_0 t} u(t) \xleftrightarrow{LT} \frac{1}{s-j\omega_0} \qquad \text{ROC} : \sigma > 0$$

Apply L.P

$$L[x(t)] = \frac{1}{2j} L\left[e^{j\omega_0 t} u(t)\right] - \frac{1}{2j} L\left[e^{-j\omega_0 t} u(t)\right]$$

$$= \frac{1}{2j}\left[\frac{1}{s-j\omega_0}\right] - \frac{1}{2j}\left[\frac{1}{s+j\omega_0}\right] \qquad \text{ROC} : \sigma > 0$$

$$X(s) = \frac{\omega_0}{s^2+\omega_0^2}$$



$$\boxed{\sin(\omega_0 t)\, u(t) \xleftrightarrow{LT} \frac{\omega_0}{s^2+\omega_0^2}}$$

ROC : $\sigma > 0$

2, Time scaling property

If $x(t) \xleftrightarrow{LT} X(s)$, ROC :- R

then $x(at) \xleftrightarrow{LT} \frac{1}{|a|} \cdot X\left(\frac{s}{a}\right)$ ROC : aR

Proof:-

i, $a > 0$

$$L[x(at)] = \int_{-\infty}^{\infty} x(at) \cdot e^{-st} dt$$

Let $at = t_1 \Rightarrow t = \frac{t_1}{a}$

$dt = \frac{1}{a} dt_1$

$$= \int_{-\infty}^{\infty} x(t_1) \cdot e^{-s\left(\frac{t_1}{a}\right)} \cdot \frac{1}{a} \cdot dt_1$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(t_1) \cdot e^{-\left(\frac{s}{a}\right) t_1} \cdot dt_1$$

$t_1 \rightarrow t$

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(t) \cdot e^{-\left(\frac{s}{a}\right) t} dt$$

$$\boxed{L[x(at)] = \frac{1}{a} X\left(\frac{s}{a}\right)}$$

ii, $a < 0$

$$\boxed{x(at) \xleftrightarrow{LT} \frac{1}{|a|} X\left(\frac{s}{a}\right)}$$

3, Time Reversal property :-

If $x(t) \xleftrightarrow{LT} X(s)$, ROC : R then $x(-t) \xleftrightarrow{LT} X(-s)$, ROC: $-R$

Proof :-

Time scaling prop

$$x(at) \xleftrightarrow{LT} \frac{1}{|a|} \cdot X\left(\frac{s}{a}\right) \quad , \text{ROC} : aR$$

Put $a = -1$

$$x(-t) \xleftrightarrow{LT} \frac{1}{|-1|} \cdot X\left(\frac{s}{-1}\right) \quad , \text{ROC} : -R$$

$$\boxed{x(-t) \xleftrightarrow{LT} X(-s)}$$

Q, find L.T of signal $x(t) = u(-t)$

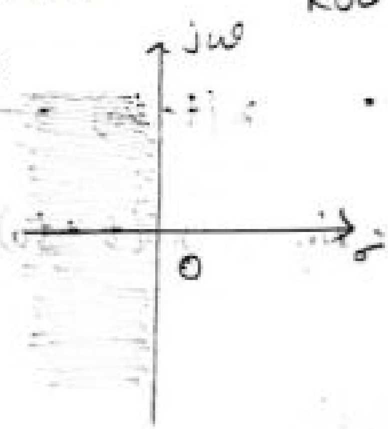
$y(t) = u(t) \xleftrightarrow{LT} Y(s) = \frac{1}{s}$; ROC : $\sigma > 0$

Apply T.R.P.

$y(-t) \xleftrightarrow{LT} Y(-s)$

$u(-t) \xleftrightarrow{LT} -\frac{1}{s}$ ROC : $\sigma < 0$

ROC



4, Time shifting property :-

If $x(t) \xleftrightarrow{LT} X(s)$, ROC: R then $x(t-t_0) \xleftrightarrow{LT} e^{-st_0} \cdot X(s)$

ROC : R

Proof :-

$$L\left[x(t-t_0)\right] = \int_{-\infty}^{\infty} x(t-t_0)\, e^{-st}\, dt$$

let $t - t_0 = t_1 \Rightarrow t = t_1 + t_0$

$dt = dt_1$

$$= \int_{-\infty}^{\infty} x(t_1) \cdot e^{-s(t_1+t_0)} dt_1$$

$$= e^{-st_0} \int_{-\infty}^{\infty} x(t_1) \cdot e^{-st_1} dt_1$$

$t_1 \rightarrow t_1$

$$= e^{-st_0} \int_{-\infty}^{\infty} x(t) \cdot e^{-st} dt$$

$$= e^{-st_0} X(s)$$

$L[x(t-t_0)] = e^{-st_0} X(s)$

- $\boxed{x(t-t_0) \xleftrightarrow{LT} e^{-st_0} X(s)}$ ROC doesn't exist

- llly, $\boxed{x(t+t_0) \xleftrightarrow{LT} e^{st_0} X(s)}$

Eg:-

1> $x(t) = \delta(t-t_0)$

$y(t) = \delta(t) \xleftrightarrow{LT} Y(s) = 1$ ROC :- entire s-plane

Apply t.s.p

$\delta(t-t_0) \xleftrightarrow{LT} e^{-st_0}$ ROC:- entire s-plane

2, $x(t) = u(t-2)$

$$y(t) = u(t) \xleftrightarrow{LT} Y(s) = \frac{1}{s} \quad ROC: \sigma > 0$$

time shifting prop

$$u(t-2) \xleftrightarrow{LT} \frac{e^{-2s}}{s} \quad Roc: \sigma > 0$$

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5, shifting in s-domain property :-

If $x(t) \xleftrightarrow{LT} X(s)$ ROC : R then

$$e^{-s_0 t} x(t) \xleftrightarrow{LT} X(s+s_0) \quad ROC: R + Re\{-s_0\}$$

Proof :-

$$L[e^{-s_0 t} x(t)] = \int_{-\infty}^{\infty} e^{-s_0 t} x(t) e^{-st} dt$$

$$= \int_{-\infty}^{\infty} x(t) e^{-(s+s_0)t} dt$$

$$= X(s+s_0)$$

$$\boxed{e^{-s_0 t} x(t) \xleftrightarrow{LT} X(s+s_0)}$$

lly, $\boxed{e^{s_0 t} x(t) \xleftrightarrow{LT} X(s-s_0)}$

$$\Rightarrow e^{-at} x(t) \xleftrightarrow{LT} X(s+a)$$

Eg:- 1, $x(t) = e^{-at} \cos(\omega_0 t) u(t)$

$$y(t) = \cos(\omega_0 t) u(t) \xleftrightarrow{LT} Y(s) = \frac{s}{s^2 + \omega_0^2} \quad ROC: \sigma > 0$$

$\sigma > 0 - a$

Apply s.s-domain prop

ROC : $\sigma > -a$

$$e^{-at} \cos(\omega_0 t) u(t) \longleftrightarrow Y(s+a) = \frac{s+a}{(s+a)^2 + \omega_0^2}$$

Roc

$$e^{-at}\cos(\omega_0 t)u(t) \xleftrightarrow{LT} \frac{s+a}{(s+a)^2+\omega_0^2} \qquad \text{Roc: } \sigma > 0-a$$

$$\sigma > -a$$

2) $x(t) = e^{-at}\sin(\omega_0 t)u(t)$.

$$\sin(\omega_0 t)u(t) \xleftrightarrow{LT} \frac{\omega_0}{s^2+\omega_0^2}$$

s.s.p

$$e^{-at}\sin(\omega_0 t)u(t) \xleftrightarrow{LT} \frac{\omega_0}{(s+a)^2+\omega_0^2} \qquad \text{Roc: } \sigma > -a$$

IMP

6) Differentiation in time domain property

If $x(t) \xleftrightarrow{LT} X(s)$ Roc : R then $\frac{dx(t)}{dt} \xleftrightarrow{LT} sX(s)$

Roc ⊃ R

Roc contains the region R

Proof :-

ILT

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,ds$$

diff w.r.t t on b.s

$$\frac{dx(t)}{dt} = \frac{1}{2\pi j}\frac{d}{dt}\left[\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\,e^{st}\,ds\right]$$

$$= \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\frac{d}{dt}[e^{st}]\,ds$$

$$= \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\,e^{st}\,s\,ds$$

$$= \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} s\, X(s) e^{st} ds$$

$$\frac{dx(t)}{dt} = \text{ILT} [s X(s)]$$

$$L\left[\frac{dx(t)}{dt}\right] = s X(s)$$

$$\boxed{\frac{dx(t)}{dt} \xleftrightarrow{LT} s X(s)}$$

lly, $\boxed{\frac{d^n x(t)}{dt^n} \xleftrightarrow{LT} s^n X(s)}$

Q, solve the following diff eqn $\frac{d^2y(t)}{dt} + y(t) = \delta(t)$

$$L\left[\frac{d^2y(t)}{dt}\right] + L[y(t)] = L[\delta(t)]$$

$$s^2 Y(s) + Y(s) = 1$$

$$Y(s) = \frac{1}{s^2+1}$$

$$\sin(\omega_0 t)\, u(t) \xleftrightarrow{LT} \frac{\omega_0}{s^2+\omega_0^2}$$

$\omega_0 = 1 \Rightarrow \sin(t)\, u(t) \xleftrightarrow{LT} \frac{1}{s^2+1}$

$\therefore$ solution for given diff eqn is $\sin t\, u(t)$

$$y(t) = \sin t\, u(t)$$

IMP Ques

Q, $$\frac{d^2y(t)}{dt} + 3\frac{dy(t)}{dt} + 2\,y(t) = \delta(t)$$

$$L\left[\frac{d^2y(t)}{dt}\right] + 3L\left[\frac{dy(t)}{dt}\right] + 2\,L[y(t)] = L[\delta(t)]$$

$$s^2Y(s) + 3s\,Y(s) + 2\,Y(s) = 1$$

$$Y(s)\left[s^2+3s+2\right] = 1$$

$$Y(s) = \frac{1}{s^2+3s+2} = \frac{1}{(s+1)(s+2)}.$$

$$Y(s) = \frac{C_1}{s+1} + \frac{C_2}{s+2}$$

By partial fractions $\Rightarrow C_1 = 1,\ C_2 = -1$

$$Y(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

ILT $\Rightarrow$ $\boxed{y(t) = e^{-t}u(t) - e^{-2t}u(t)}$

7) Differentiation in s-domain property :-

If $x(t) \xleftrightarrow{LT} X(s)$ ROC: R then $-t\,x(t) \xleftrightarrow{LT} \frac{dX(s)}{dt}$

ROC $\supseteq$ R

Proof:-

$$L[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t)\,e^{-st}\,dt.$$

$$\frac{dX(s)}{ds} = \int_{-\infty}^{\infty} x(t)\,\frac{d}{ds}\left(e^{-st}\right)dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-st} (-t)\, dt$$

$$= \int_{-\infty}^{\infty} [-t\, x(t)]\, e^{-st}\, dt$$

$$\frac{d\,X(s)}{ds} = L\left[-t\, x(t)\right]$$

$$\boxed{-t\, x(t) \xleftrightarrow{LT} \frac{d\,X(s)}{ds}}$$

Illy, $\boxed{t^n\, x(t) \xleftrightarrow{LT} \frac{d^n X(s)}{ds^n}}$

Ex:- Q, $x(t) = t\, e^{-at} u(t)$

$$e^{-at} u(t) \xleftrightarrow{LT} \frac{1}{s+a}$$

Apply d.s P

$$-t\, e^{-at} u(t) \xleftrightarrow{LT} \frac{d}{ds}\left(\frac{1}{s+a}\right)$$

$$\cancel{-}t\, e^{-at} u(t) \xleftrightarrow{LT} \cancel{-}\frac{1}{(s+a)^2} \qquad ROC: \sigma > -a$$

Q, Find the L.T of Ramp fn signal $x(t) = t \cdot u(t)$

$$y(t) = t\, e^{-at}\, u(t) \xleftrightarrow{LT} Y(s) = \frac{1}{(s+a)^2} \qquad ROC: \sigma > -a$$

Put $a = 0 \Rightarrow$ $t\, u(t) \xleftrightarrow{LT} \frac{1}{s^2}$ $\qquad ROC: \sigma > 0$

Q, $x(t) = t^2 u(t)$

$$t\,u(t) \xleftrightarrow{LT} \frac{1}{s^2}$$

d·s·p

$$-t\,t u(t) \xleftrightarrow{LT} \frac{d}{ds}\left[\frac{1}{s^2}\right]$$

$$\cancel{-}t^2 u(t) \longleftrightarrow \cancel{-}\frac{2}{s^3}$$

$$\therefore \boxed{t^2 u(t) \xleftrightarrow{LT} \frac{2}{s^3}} \quad \text{. Lly, } t^3 u(t) \xleftrightarrow{LT} \frac{6}{s^4}$$

$$\therefore \text{ for } n \Rightarrow \boxed{\frac{t^n}{n!} u(t) \xleftrightarrow{LT} \frac{1}{s^{n+1}}} \qquad \text{ROC} : \sigma > 0$$

Q, $x(t) = \frac{t^n}{n!} e^{-at} \cdot u(t)$

$$\frac{t^n}{n!} u(t) \xleftrightarrow{LT} \frac{1}{s^{n+1}}$$

shifting in s-domain

$$\frac{t^n}{n!} e^{-at} u(t) \xleftrightarrow{LT} \frac{1}{(s+a)^{n+1}}$$

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8, convolution theorem :-

If $x_1(t) \xleftrightarrow{LT} X_1(s)$ ROC : R_1 , $x_2(t) \xleftrightarrow{LT} X_2(s)$ ROC : R_2

then $x_1(t) * x_2(t) \xleftrightarrow{LT} X_1(s)\, X_2(s)$ ROC : $R_1 \cap R_2$

Proof:-

$$L[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} [x_1(t) * x_2(t)] e^{-st} dt.$$

w.k.T $x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau) d\tau$

$$\Rightarrow L[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau) d\tau\right] e^{-st} dt$$

$$= \int_{-\infty}^{\infty} x_1(\tau) \underbrace{\left[\int_{-\infty}^{\infty} x_2(t-\tau) e^{-st} dt\right]}_{L[x_2(t-\tau)] = e^{-s\tau} X_2(s)} d\tau$$

$$= \int_{-\infty}^{\infty} x_1(\tau) X_2(s) e^{-s\tau} d\tau$$

$$= X_2(s) \underbrace{\int_{-\infty}^{\infty} x_1(\tau) e^{-s\tau} d\tau}_{L[x_1(t)] = X_1(s)}$$

$$= X_2(s) X_1(s)$$

$$\boxed{x_1(t) * x_2(t) \xleftrightarrow{LT} X_1(s)\, X_2(s)}$$

- $x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau) d\tau = L^{-1}[X_1(s) X_2(s)]$

- $L[x_1(t) * x_2(t)] = L[x_1(t)] \cdot L[x_2(t)]$

Q, Find the convolution b/w 2 signals :-

1, $x_1(t) = u(t)$, $x_2(t) = u(t)$

$x_1(t) = u(t) \xleftrightarrow{LT} X_1(s) = \frac{1}{s}$ ROC : $\sigma > 0$

$x_2(t) = u(t) \xleftrightarrow{LT} X_2(s) = \frac{1}{s}$ ROC : $\sigma > 0$

$x_1(t) * x_2(t) = L^{-1}[X_1(s) X_2(s)] = \underbrace{L^{-1}\left(\frac{1}{s^2}\right)}_{\text{Ramp signal}}$ ROC : $\sigma > 0$

2) $x_1(t) = e^{-at}u(t)$, $x_2(t) = e^{-bt}u(t)$

$$x_1(t) = e^{-at}u(t) \xrightarrow{LT} X_1(s) = \frac{1}{s+a} \quad ROC: \sigma > -a$$

$$x_2(t) = e^{-bt}u(t) \xrightarrow{LT} X_2(s) = \frac{1}{s+b} \quad ROC: \sigma > -b$$

$$x_1(t) * x_2(t) = L^{-1}\left[\frac{1}{s+a} \cdot \frac{1}{s+b}\right]$$

$$L\left[x_1(t) * x_2(t)\right] = \frac{1}{s+a} \cdot \frac{1}{s+b}$$

$$\Rightarrow X(s) = \frac{1}{(s+a)\cdot(s+b)} = \frac{C_1}{s+a} + \frac{C_2}{s+b}$$

$$1 = C_1(s+b) + C_2(s+a).$$

If $s = -b \Rightarrow \quad C_2 = \frac{1}{a-b}$

$s = -a \Rightarrow \quad C_1 = -\frac{1}{a-b}$

$$X(s) = -\frac{1}{a-b}\left[\frac{1}{s+a} - \frac{1}{s+b}\right]$$

$$x(t) = \frac{e^{-at}u(t) - e^{-bt}u(t)}{b-a}$$

9, Multiplication theorem [convolution in s-domain]:-

if $x_1(t) \xleftrightarrow{LT} X(s)$ ROC: R_1 , $x_2(t) \xleftrightarrow{LT} X_2(s)$ ROC: R_2

then $x_1(t) \cdot x_2(t) \xleftrightarrow{LT} \frac{1}{2\pi j}[X_1(s) * X_2(s)]$ ROC: $R_1 \cap R_2$

Proof :-

$$ILT \Rightarrow x_1(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X_1(s)\, e^{st}\, ds$$

$$L[x_1(t)\cdot x_2(t)] = \int_{-\infty}^{\infty} [x_1(t)\cdot x_2(t)]\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} \left[\frac{1}{2\pi j}\int_{\sigma-j\omega}^{\sigma+j\omega} X_1(\lambda)\, e^{\lambda t}\, d\lambda\right] x_2(t) e^{-st}\, dt$$

$$= \frac{1}{2\pi j}\int_{-\infty}^{\infty} X_1(\lambda)\left[\underbrace{\int_{-\infty}^{\infty} x_2(t)\, e^{-(s-\lambda)}\, dt}_{X_2(s-\lambda)}\right] d\lambda$$

$$= \frac{1}{2\pi j}\int_{-\infty}^{\infty} X_1(\lambda)\, X_2(s-\lambda)\, d\lambda$$

$$= \frac{1}{2\pi j}[X_1(s) * X_2(s)]$$

$$\boxed{x_1(t)\cdot x_2(t) \xleftrightarrow{LT} \frac{1}{2\pi j}[X_1(s) * X_2(s)]}$$

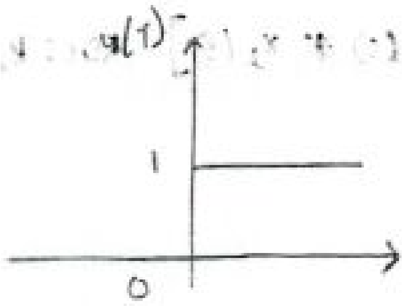
10, Integration in time domain property:-

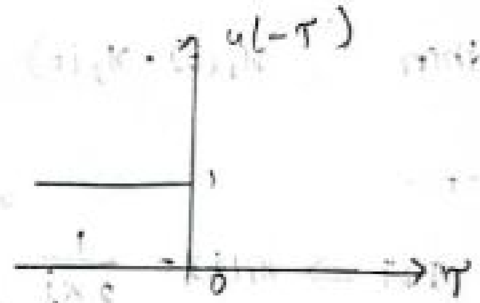
If $x(t) \xleftrightarrow{LT} X(s)$ ROC: R then

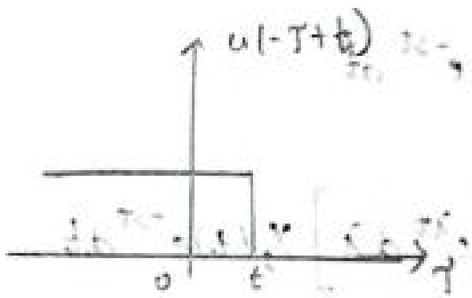
$$\int_{-\infty}^{t} x(\tau)\, d\tau \xleftrightarrow{LT} \frac{X(s)}{s}$$ ROC :- $R \cap Re(s) > 0$

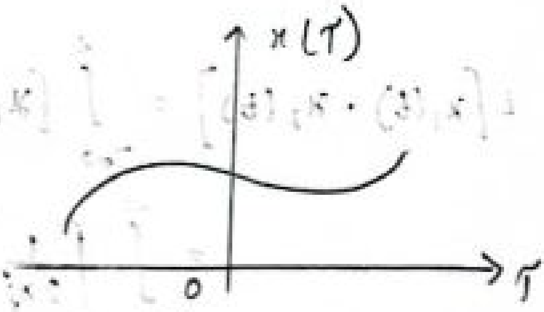
Proof

$$x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau)\, u(t-\tau)\, d\tau$$









$$x(t) * u(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

$$L[x(t) * u(t)] = L\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right]$$

$$L\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right] = L\{x(t)\} \cdot L[u(t)]$$

$$= X(s) \cdot \frac{1}{s}$$

$$\boxed{\int_{-\infty}^{t} x(\tau) \cdot d\tau \xleftrightarrow{LT} \frac{X(s)}{s}}$$

Q) $\int_{0}^{t} y(t) \cdot dt = 3 \cdot u(t) - 2 \cdot y(t)$. Find $y(t)$ -

Apply L·T on b/s

$$\frac{Y(s)}{s} = \frac{3}{s} - 2 \cdot Y(s)$$

$$Y(s)\left[\frac{1}{s} + 2\right] = 3/s$$

$$Y(s)\left[\frac{1+2s}{s}\right] = \frac{3}{8}$$

$$Y(s) = \frac{3}{1+2s} = \frac{3/2}{s+1/2}$$

$$y(t) = \frac{3}{2}e^{-\frac{1}{2}t}u(t)$$

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11) Integration in s-domain property :-

If $x(t) \xleftrightarrow{LT} X(s)$ ROC : R then $\frac{x(t)}{t} \xleftrightarrow{LT} \int_s^{\infty} X(s)\,ds$

ROC : R

Proof:-

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$\int_s^{\infty} X(s)\,ds = \int_s^{\infty}\left[\int_{-\infty}^{\infty} x(t)e^{-st}dt\right]ds$$

$$= \int_{-\infty}^{\infty}\left[x(t)\int_s^{\infty} e^{-st}ds\right]dt$$

$$\Rightarrow \int_s^{\infty} e^{-st}ds = \left[\frac{e^{-st}}{-t}\right]_s^{\infty} = -\frac{1}{t}\left[e^{-\infty t} - e^{-st}\right] = \frac{1}{t}e^{-st}$$

$$\int_s^{\infty} X(s)\,ds = L\left[\frac{x(t)}{t}\right]$$

$$\boxed{\frac{x(t)}{t} \xleftrightarrow{LT} \int_s^{\infty} X(s)\,ds}$$

Q) Find the LT of signal $x(t) = \frac{\sin(\omega t)u(t)}{t}$

$y(t) = \sin(\omega t) \xleftrightarrow{LT} Y(s) = \frac{\omega}{s^2+\omega^2}$ ROC : $\sigma > 0$

Apply Inte in

$$L\left[\frac{\sin(\omega t)}{t}u(t)\right] = \int_s^\infty \frac{\omega}{s^2+\omega^2}$$

$$= \left[\frac{1}{\omega} \cdot \omega \cdot \tan^{-1}\left(\frac{s}{\omega}\right)\right]_s^\infty$$

$$= \tan^{-1}(\infty) - \tan^{-1}\left(\frac{s}{\omega}\right)$$

$$= \frac{\pi}{2} - \tan^{-1}(s/\omega)$$

$$= \cot^{-1}(s/\omega)$$

Inverse Laplace transform Calculations :-

• Calculate the ILT for the following

1) $X(s) = \frac{2s+4}{s^2+4s+3}$

a) $Re(s) > -1$ causal

b) $Re(s) < -3$ anticausal

c) $-3 < Re(s) < -1$ noncausal

$$X(s) = \frac{2s+4}{(s+1)(s+3)} = \frac{C_1}{s+1} + \frac{C_2}{s+3}$$

$$2s+4 = C_1(s+3) + C_2(s+1)$$

If $s = -3 \Rightarrow -6+4 = -2C_2 \Rightarrow C_2 = 1.$

If $s = -1 \Rightarrow 2 = 2C_1 \Rightarrow C_1 = 1$

other method :- $C_1 = (s+1)X(s)\big|_{s=-1}$ $C_2 = (s+3)X(s)\big|_{s=-3}$

$$X(s) = \frac{1}{s+1} + \frac{1}{s+3}$$

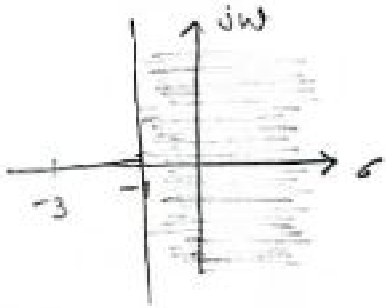
a) $Re(s) > -1$

$\sigma > -1$

The signal is causal

[max]

$x(t) = e^{-t}u(t) + e^{-3t}u(t)$



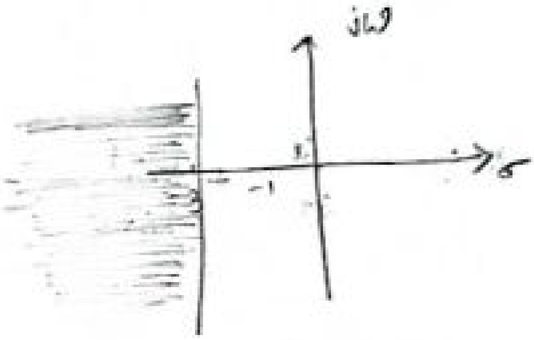
b) $Re(s) < -3$

$\sigma < -3$

Anti-causal

$e^{-t}u(-t) \overset{L}{\longleftrightarrow} \frac{-1}{s+1} \quad \sigma < -1$

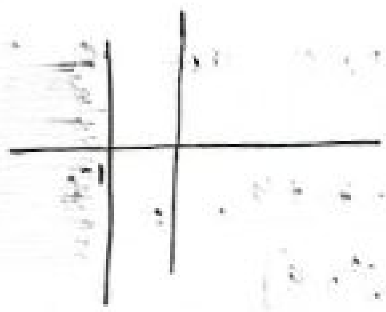
$x(t) = -e^{-t}u(-t) - e^{-3t}u(-t)$



c) $-3 < Re(s) < -1$

$Re(s) > -3$ $\qquad$ $Re(s) < -1$

- The pole corresponding to -3 is causal
- The pole corresponding to -1 is anti causal signal



ii) $X(s) = \dfrac{s^2 + 2s + 5}{(s+3)(s+5)^2}$

a) $Re(s) > -3$

$$x(s) = \frac{s^2+2s+5}{(s+3)(s+5)^2} = \frac{c_1}{s+3} + \frac{\lambda_1}{s+5} + \frac{\lambda_2}{(s+5)^2}$$

$$s^2+2s+5 = c_1(s+5)^2 + \lambda_1(s+3)(s+5) + \lambda_2(s+3)$$

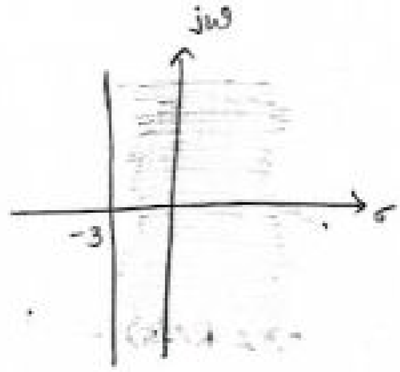
If $s=-5 \Rightarrow \overset{10}{\cancel{20}} = -\cancel{2}\lambda_2 \Rightarrow \lambda_2 = -10$

$s=-3 \Rightarrow 8 = 4c_1 \Rightarrow c_1 = 2$

$s^2 \Rightarrow 1 = c_1 + \lambda_1 \Rightarrow \lambda_1 = -1$

$$x(s) = \frac{2}{s+3} + \frac{-1}{s+5} - \frac{10}{(s+5)^2}$$

$Re(s) > -3 \Rightarrow$ causal causal causal



$$x(t) = 2e^{-3t}u(t) - e^{-5t}u(t) - 10te^{-5t}u(t)$$

3) $x(s) = \dfrac{3s^2+7s+4}{s[(s+2)^2+9]}$ $\qquad s^3+13s+4s^2$

The poles are $0, -2\pm 3j$

$$\frac{3s^2+7s+4}{s[(s+2)^2+9]} = \frac{A}{s} + \frac{Bs+C}{(s+2)^2+9}$$

$$3s^2+7s+4 = A((s+2)^2+9) + Bs^2 + Cs$$

$$(A+B)s^2 + 13A + (4A+C)s$$

$13A = 4 \Rightarrow A = \frac{4}{13}$, $\frac{16}{13} + C = 7 \Rightarrow C = 7 - \frac{16}{13} = \frac{75}{13}$

$$\frac{4}{3} + B = 3$$

$$B = 3 - \frac{4}{13} = \frac{35}{13}$$

$$X(s) = \frac{4}{13s} + \frac{\frac{35}{13}s + \frac{75}{13}}{(s+2)^2 + 3^2}$$

$$x(t) = \frac{4}{13}\left(\frac{1}{s}\right) + \frac{\frac{35}{13}(s+2-2)}{(s+2)^2+3^2} + \frac{75}{13}\left[\frac{1}{(s+2)^2+3^2}\right]$$

$$= \frac{4}{13}\left(\frac{1}{s}\right) + \frac{35}{13}\left[\frac{s+2}{(s+2)^2+3^2}\right] + \frac{1}{(s+2)^2+3^2}\left[\frac{75}{13} - \frac{70}{13}\right]$$

$$= \frac{4}{13}\left(\frac{1}{s}\right) + \frac{35}{13}\left[\frac{s+2}{(s+2)^2+3^2}\right] + \frac{5}{13 \times 3}\left[\frac{3}{(s+2)^2+3^2}\right]$$

ILT

$$x(t) = \frac{4}{13}u(t) + \frac{35}{13} \cdot e^{-2t}\cos(3t) \cdot u(t) + \frac{5}{39}e^{-3t}\sin(3t)\,u(t).$$

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4> $$X(s) = \frac{s^3+1}{s(s+1)(s+2)}$$

$$s^3 + 3s^2 + 2s \overline{)\; s^3 + 1 \;(} 1$$
$$\underline{-\; s^3 + 3s^2 + 2s}$$
$$1 - 3s^2 + 2s$$

$$\frac{s^3+1}{s^3+3s^2+2s} = 1 + \frac{1-3s^2-2s}{s^3+3s^2+2s}$$

$$x(s) = 1 - \left[\frac{3s^2+2s-1}{s(s+1)(s+2)}\right]$$

$x(s) = 1 - X_1 s$ where $X_1(s) = \dfrac{3s^2+2s-1}{s(s+1)(s+2)}$

$$X_1(s) = \frac{c_1}{s} + \frac{c_2}{s+1} + \frac{c_3}{s+2}$$

$3s^2+2s-1 = c_1(s+1)(s+2) + c_2(s+2)(s) + c_3(s)(s+1)$

If $s=0 \Rightarrow -1 = 2c_1 \Rightarrow c_1 = -1/2$

$s=-1 \Rightarrow 0 = -1c_2 \Rightarrow c_2 = 0$

$s=-2 \Rightarrow 7 = 2c_3 \Rightarrow c_3 = 7/2$

$$X_1(s) = -\frac{1}{2}\left(\frac{1}{s}\right) + 0 + \frac{7}{2}\left(\frac{1}{s+2}\right)$$

$$x(s) = 1 - X_1(s)$$

$$= 1 + \frac{1}{2}\left(\frac{1}{s}\right) - \frac{7}{2}\left(\frac{1}{s+2}\right)$$

If ROC is not given then choose causal

$$x(t) = \delta(t) + \frac{1}{2}u(t) - \frac{7}{2}e^{-2t}u(t)$$

5) $x(s) = \dfrac{1+e^{-2s}}{3s^2+2s}$

let $X_1(s) = \dfrac{1}{3s^2+2s}$

$$x(s) = (1+e^{-2s})X_1(s)$$

$$x_1(s) = \frac{1}{3s^2+2s} = \frac{c_1}{s} + \frac{c_2}{3s+2}$$

$$1 = c_1(3s+2) + c_2(s)$$

If $s=0 \Rightarrow c_1 = 1/2$

$s = -\frac{2}{3} \Rightarrow c_2 = -3/2$

$$x_1(s) = \frac{1}{2}\left(\frac{1}{s}\right) + \left(-\frac{3}{2}\right)\left(\frac{1}{3s+2}\right)$$

mul & div 3

$$= \frac{1}{2}\left(\frac{1}{s}\right) - \frac{1}{2}\left(\frac{1}{s+2/3}\right)$$

from ①,

$$x(s) = x_1(s) + e^{-2s} x_1(s)$$

ILT

$$x(t) = x_1(t) + x_1(t-2)$$

$$= \frac{1}{2}u(t) - \frac{1}{2}e^{-2/3\,t}u(t)$$

b) $X(s) = \frac{1}{(s+1)(s^2+1)}$ using convolution theorem.

$$X(s) = X_1(s) \cdot X_2(s)$$

$$X_1(s) = \frac{1}{s+1}, \quad X_2(s) = \frac{1}{s^2+1}$$

ILT

$$L^{-1}[X(s)] = L^{-1}[X_1(s) \cdot X_2(s)]$$

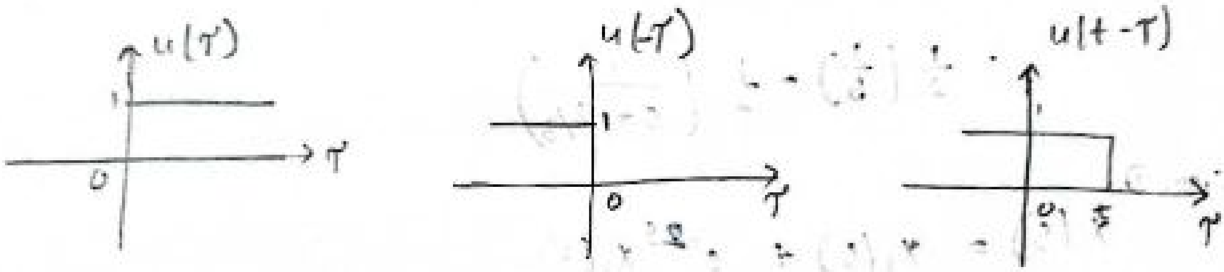
$$x(t) = x_1(t) * x_2(t)$$

$x_1(t) = e^{-t}u(t) \xleftrightarrow{LT} X_1(s) = \frac{1}{s+1}$

$x_2(t) = \sin t \cdot u(t) \xleftrightarrow{LT} X_2(s) = \frac{1}{s^2+1}$

$$x(t) = x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\cdot x_2(t-\tau)\,d\tau = \int_{-\infty}^{\infty} x_2(\tau)\,x_1(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} \sin(\tau)\,u(\tau)\cdot e^{-(t-\tau)}\,u(t-\tau)$$



$$x(t) = \int_0^t e^{-(t-\tau)}\sin(\tau)\,d\tau$$

$$= e^{-t}\int_0^t e^{\tau}\sin\tau\,d\tau \qquad \int e^{ax}\sin(bx)\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\sin(bx) - b\cos(bx)\right)$$

$$= e^{-t}\left[\frac{e^{\tau}}{1+1}(\sin\tau - \cos\tau)\right]_0^t$$

$$= e^{-t}\left[\frac{e^t}{2}(\sin t - \cos t) - \frac{e^0}{2}(-1)\right]$$

$$= \frac{e^{-t}}{2}\left[e^t(\sin t - \cos t) + 1\right]$$

$$= \frac{1}{2}\left[\sin t - \cos t + e^{-t}\right] \xRightarrow{L.T} \left[\begin{array}{l} \frac{1}{2}\left[\frac{1}{s^2+1} - \frac{s}{s^2+1} + \frac{1}{s+1}\right] \\ = \frac{1}{2}\left[\frac{1-s}{s^2+1} + \frac{1}{s+1}\right] = \frac{1}{(s^2+1)(s+1)} \end{array}\right.$$

$$x(t) = \frac{1}{2}\left[\sin t\,(u(t)) - \cos t\,(u(t))\right] + \frac{1}{2}e^{-t}u(t)\,.$$

7) $X(s) = \dfrac{1}{s(s^2+1)}$, using convolution

Applications of L.T :-

↳ Poles and zeros of a system :-

$$x(t) \longrightarrow \boxed{T(\cdot)} \longrightarrow y(t)$$

LTI sys (continuous

· Any continuous time sys can always be represented by its integro-differential eqn

$$y(t) = -\sum_{K=1}^{N} a_K \frac{d^K y(t)}{dt^K} + \sum_{K=0}^{M} b_K \frac{d^K x(t)}{dt^K}$$

Apply L.T

$$L[y(t)] = -\sum_{K=1}^{N} a_K \, L\left[\frac{d^K y(t)}{dt^K}\right] + \sum_{K=0}^{M} b_K \, L\left[\frac{d^K x(t)}{dt^K}\right]$$

$$Y(s) = -\sum_{K=1}^{N} a_K s^K Y(s) + \sum_{K=0}^{M} b_K s^K X(s)$$

$$Y(s) + \sum_{K=1}^{N} a_K s^K Y(s) = \sum_{K=0}^{M} b_K s^K X(s)$$

$$Y(s)\left[1 + \sum_{K=1}^{N} a_K s^K\right] = \sum_{K=0}^{M} b_K s^K X(s)$$

$$\boxed{H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{K=0}^{M} b_K s^K}{1 + \sum_{K=1}^{N} a_K s^K}}$$

H(s) is the transfer function of the system

$N \geq M \Rightarrow$ where capital is called order of system

$$\boxed{H(s) = \frac{b_0 + b_1 s + b_2 s^2 + \cdots + b_M s^M}{1 + a_1 s + a_2 s^2 + \cdots + a_N s^N}} \Rightarrow \text{Transfer function}$$

→ zeros

→ poles

- zeros of the system can be computed by equating numerater polynomial of $H(s) = 0$
- Zeros are always represented by 'o' [circle] is s-plane
- At zero location the value of the TF $H(s) = 0$

- Poles can be computed by equating denominator polynomial of $H(s) = 0$
- Poles are denoted by 'x' in s-plane
- At pole location value of TF $H(s) = \infty$

1/11

2, Impulse response & T.F of the System

$$x(t) \xrightarrow{i/p} \boxed{h(t)} \xrightarrow{o/p} y(t)$$

LTI system

$$y(t) = x(t) * h(t)$$

$$L[y(t)] = L[x(t) * h(t)]$$

Apply convolution theorem

$$Y(s) = L[x(t)]\, L[h(t)]$$

$$\boxed{Y(s) = X(s)\,H(s)} \rightarrow \text{o/p of LTI system.}$$

$$H(s) = \frac{Y(s)}{X(s)} \longrightarrow \text{T.F (or) system fun}$$

$$H(s) = L[h(t)]$$

$$\boxed{h(t) \xleftrightarrow{LT} H(s)}$$

$h(t)$ → Impulse response of sys

$H(s)$ → TF of system

Q) i/p to the LTI system is $x(t) = e^{-2t}u(t)$ & the impulse response of the system is $h(t) = e^{-3t}u(t)$. Find the o/p of the system.

i/p, IR, o/p

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = x(t) * h(t)$$

(or)

T.F

$$X(s) \longrightarrow \boxed{H(s)} \longrightarrow Y(s) = X(s)\cdot H(s)$$

$$y(t) = L^{-1}[Y(s)]$$

$$x(t) = e^{-2t}u(t) \xleftrightarrow{LT} X(s) = \frac{1}{s+2}$$

$$h(t) = e^{-3t}u(t) \xleftrightarrow{LT} H(s) = \frac{1}{s+3}$$

$$Y(s) = X(s)\cdot H(s) = \frac{1}{(s+2)(s+3)} = \frac{c_1}{s+2} + \frac{c_2}{s+3}$$

If $s = -2 \Rightarrow c_1 = 1$; If $s = -3 \Rightarrow c_2 = -1$

$$Y(s) = \frac{1}{s+2} + \frac{-1}{s+3}$$

$$y(t) = e^{-2t}u(t) - e^{-3t}u(t)$$

Q, The o/p of the system is $y(t) = 2 \cdot e^{-3t}u(t)$. when the i/p $x(t) = u(t)$. find IR of system & the o/p when i/p is $e^{-t}u(t)$ -

$$\begin{array}{c} \text{i/p} \\ x(t) \\ = u(t) \end{array} \longrightarrow \boxed{h(t)} \xrightarrow{\text{o/p}} y(t) = 2e^{-3t}u(t)$$

$$Y(s) = X(s)\,H(s)$$

TF
$$H(s) = \frac{Y(s)}{X(s)}$$

$$x(t) = u(t) \xleftrightarrow{LT} X(s) = \frac{1}{s}$$

$$y(t) = 2e^{-3t}u(t) \xleftrightarrow{LT} Y(s) = \frac{2}{s+3}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{2s}{s+3} = \frac{2[s+3-3]}{s+3} = 2\left[1 - \frac{3}{s+3}\right]$$

$$H(t) = L^{-}[H(s)] = L^{-}\left[2\left[1 - \frac{3}{s+3}\right]\right]$$

$$= 2\left[\delta(t) - 3e^{-3t}u(t)\right]$$

3, Step Response of the system

$$\underset{x(t)=u(t)}{\overset{i/p}{}} \longrightarrow \overset{\text{LTI system}}{\boxed{h(t)}} \longrightarrow \overset{o/p}{y(t)=s(t)}$$

diff - Im

Inte - stp

o/p of the system is $Y(s) = X(s)\,H(s)$

$$x(t) = u(t) \xleftrightarrow{LT} X(s) = \frac{1}{s}$$

$$Y(s) = \frac{H(s)}{s}$$

$$y(t) = L^{-1}[Y(s)] = L^{-1}\left[\frac{H(s)}{s}\right] = \int_{-\infty}^{t} h(t)\,dt$$

$y(t) = s(t) \longrightarrow$ step response of the system

$$\boxed{s(t) = \int_{-\infty}^{t} h(t)\,dt} \Longrightarrow \boxed{h(t) = \frac{ds(t)}{dt}}$$

Q, the step response of the system is $e^{-3t}u(t)$, then find the o/p of the system for the i/p $e^{-2t}u(t)$ -

the IR of system $h(t) = \frac{ds(t)}{dt}$

$$h(t) = e^{-3t}(-3)\,u(t) + e^{-3t}\,\delta(t)$$

$$h(t) = -3e^{-3t}\cdot u(t) + \delta(t)$$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$Y(s) = X(s)\cdot H(s)$

$$x(t) = e^{-2t}u(t) \xleftrightarrow{LT} X(s) = \frac{1}{s+2}$$

$$H(s) = \frac{-3}{s+3} + 1 = \frac{-3+s+3}{s+3}$$

$$H(s) = \frac{s}{s+3}$$

$$Y(s) = X(s) \cdot H(s) = \frac{s}{(s+2)(s+3)} = \frac{C_1}{s+2} + \frac{C_2}{s+3}$$

at $s = -2 \Rightarrow -2 = C_1$, $s = -3 \Rightarrow -3 = -C_2$

$$C_1 = -2, \ C_2 = 3$$

$$Y(s) = \frac{-2}{s+2} + \frac{3}{s+3}$$

$$\boxed{y(t) = -2 \cdot e^{-2t} \cdot u(t) + 3 \cdot e^{-3t} u(t)}$$

4) causal system :-

If Impulse response of the system $h(t)$ is 0 for $t<0$ then the system is causal system.

$$h(t) \xleftrightarrow{LT} H(s)$$

IR TF

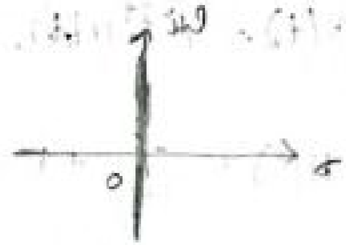
ROC of $H(s)$ is $\sigma > \sigma_{max}$

5) stable system :-

If $\int_{-\infty}^{\infty} |h(t)| dt < \infty$ then the system is stable system.

$$\underset{TF}{H(s)} = L[h(t)] = \int_{-\infty}^{\infty} h(t) e^{-st} dt$$

$s = \sigma + j\omega$



Assume $\sigma = 0 \Rightarrow s = j\omega$

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt$$

$$|H(j\omega)| = \left| \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt \right|$$

$$\leq \int_{-\infty}^{\infty} h(t)\, |e^{-j\omega t}|\, dt$$

$$\leq \int_{-\infty}^{\infty} |h(t)|\, dt$$

For stable system, $\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$

$$|H(j\omega)| < \infty \Rightarrow \boxed{|H(s)|\big|_{s=j\omega} < \infty}$$

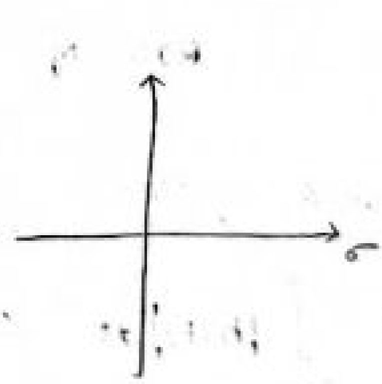
ROC of $H(s)$ must includes the "$j\omega$" axis for the stable system

11/11

6, Causal & stable system

causal :- $\sigma > \sigma_{max}$ $\begin{cases} = 0 \\ > 0 \\ < 0 \end{cases}$

stable :- includes $j\omega$ axis



ROC of $H(s)$:- $\sigma < \sigma_{max} < 0 \Rightarrow$ for causal & stable system

- All the poles of $H(s)$ must lie on the LHS of s plane for causal & stable system

Unilateral laplace Transform :- [single sided L.T]

$$x(t) \xleftrightarrow{LT} X(s)$$

ULT :- $X(s) = \int_{0}^{\infty} x(t) e^{-st} dt$ BLT :- $X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$

- If the signal is causal signal [i.e $x(t) = 0$ for $t < 0$] then it is Bilateral L.T [BLT]

$$X(s) = \int_{0}^{\infty} x(t) e^{-st} dt$$

- For causal signal, ULT = BLT ROC : $\sigma > \sigma_{max}$

same always

Properties:- [All other properties are same as BLT Expect given]

1) Differentiation property:-

If $x(t) \xleftrightarrow{LT} X(s)$ then $\frac{dx(t)}{dt} \xleftrightarrow{LT} sX(s) - \underbrace{x(0)}_{\text{initial condition}}$

Proof:-

$$LT\left[\frac{dx(t)}{dt}\right] = \int_{0}^{\infty} \left[\frac{dx(t)}{dt}\right] e^{-st} dt$$

$\int uv = u\int v - \int u' \int v \, dx$

$$= \left[e^{-st} x(t)\right]_{0}^{\infty} - \int_{0}^{\infty} (-s) e^{-st} x(t) \, dt$$

$$= \left[e^{-\infty} - e^{0} x(0)\right] + s\,X(s)$$

$$= -x(0) + sX(s)$$

$$\boxed{\frac{dx(t)}{dt} \xleftrightarrow{LT} s(X(s)) - x(0)}$$

initial condition

- $L\left[\frac{d\,x(t)}{dt}\right] = sX(s) - x(0)$

$$= s\,L[x(t)] - x(0)$$

- $L\left[\frac{d^2x(t)}{dt^2}\right] = L\left[\frac{d}{dt}\cdot\frac{d\,x(t)}{dt}\right] = s\,L\left[\frac{d\,x(t)}{dt}\right] - x'(0)$

$$= s[sX(s) - x(0^-)] - x'(0)$$

$$\boxed{L\left[\frac{d^2x(t)}{dt^2}\right] = s^2X(s) - s\,x(0^-) - x'(0^-)}$$

- $\boxed{L\left[\frac{d^3x(t)}{dt^3}\right] = s^3X(s) - s^2x(0^-) - s^1x'(0^-) - x'(0^-)}$

2) Integration property :-

If $x(t) \xleftrightarrow{LT} X(s)$ then $\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{LT} \frac{X(s)}{s} + \int_{-\infty}^{t} \frac{x(0)}{s}dt$

Proof :-

$$f(t) = \int_{-\infty}^{t} x(t)dt \quad -①$$

$$\frac{d\,f(t)}{dt} = x(t) \Longrightarrow L\left[\frac{d\,f(t)}{dt}\right] = L[x(t)]$$

$$s\,F(s) - f(0^-) = X(s)$$

$$sF(s) = X(s) + f(0^-)$$

$$F(s) = \frac{X(s)}{s} + \frac{1}{s}f(0^-)$$

$$f(0^-) = \int_{-\infty}^{0} x(0)\, dt$$

$$\boxed{L[f(t)] = F(s) = \frac{X(s)}{s} + \frac{1}{s}\int_{-\infty}^{0} x(0)\, dt}$$

3, Initial value theorem:-

If $x(t) \xleftrightarrow{LT} X(s)$ then $x(0) = \lim_{s \to \infty} s X(s)$

Proof:-

$$L\left[\frac{d\,x(t)}{dt}\right] = \int_0^{\infty} \frac{dx(t)}{dt} e^{-st} dt$$

$$sX(s) - x(0) = \int_0^{\infty} \frac{dx(t)}{dt} e^{-st} dt$$

$$\lim_{s \to \infty} [sX(s) - x(0)] = \lim_{s \to \infty} \left[\int_0^{\infty} \frac{dx(t)}{dt} e^{-st} dt\right]$$

$$\lim_{s \to \infty} sX(s) - x(0) = 0$$

$$\boxed{x(0) = \lim_{s \to \infty} sX(s)}$$

4, Final value theorem:-

If $x(t) \xleftrightarrow{LT} X(s)$ then $x(\infty) = \lim_{s \to 0} s X(s)$.

Proof:-

$$L\left[\frac{d\,x(t)}{dt}\right] = \int_0^{\infty} \frac{dx(t)}{dt} e^{-st} dt$$

$$s \cdot X(s) - x(0) = \int_0^{\infty} \frac{dx(t)}{dt} e^{-st} dt$$

$$\lim_{s \to 0} \left[sX(s) - x(0) \right] = \lim_{s \to 0} \left[\int_0^{\infty} \frac{dx(t)}{dt} e^{-st} dt \right]$$

$$= \int_0^{\infty} \frac{dx(t)}{dt} \left(\lim_{s \to 0} e^{-st} \right) dt$$

$$\lim_{s \to 0} sX(s) - \cancel{x(0)} = x(\infty) - \cancel{x(0)}$$

$$\boxed{x(\infty) = \lim_{s \to 0} sX(s)}$$

- Final value theorem is applicable all the poles of $sX(s)$ must lie on LHS of s-plane

12/11 Problems [imp 10M]

1Q, System is described by the following differential equation

i, $\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = x(t)$. Find -

a, impulse response of system

b, step response of system

sol Given $\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = x(t)$

$$L\left[\frac{d^2y(t)}{dt^2}\right] + 5L\left[\frac{dy(t)}{dt}\right] + 6L[y(t)] = L[x(t)]$$

$$\left[s^2 Y(s) - s \underbrace{y(0^-)}_{0} - \underbrace{y'(0^-)}_{0}\right] + 5\left[sY(s) - \underbrace{y(0^-)}_{0}\right] + 6Y(s) = X(s)$$

since initial conditions are not given they are zero

$$Y(s)\left[s^2 + 5s + 6\right] = X(s)$$

a) $x(t) = \delta(t) \xleftrightarrow{LT} X(s) = 1$

$$Y(s) = \frac{1}{s^2+5s+6} = \frac{1}{(s+2)(s+3)} = \frac{1}{s+2} - \frac{1}{s+3}$$

Apply ILT

$$\boxed{y(t) = e^{-2t} \cdot u(t) - e^{-3t} u(t)}$$

b) $x(t) = u(t) \xleftrightarrow{LT} X(s) = \frac{1}{s}$

$$Y(s) = \frac{1}{s(s^2+5s+6)} = \frac{1}{s(s+2)(s+3)} = \frac{C_1}{s} + \frac{C_2}{s+2} + \frac{C_3}{s+3}$$

If $s = 0 \Rightarrow 1 = 6C_1 \Rightarrow C_1 = 1/6$

$s = -2 \Rightarrow 1 = -2C_2 \Rightarrow C_2 = -1/2$

$s = -3 \Rightarrow 1 = 3C_3 \Rightarrow C_3 = 1/3$

$$Y(s) = \frac{1}{6}\left(\frac{1}{s}\right) - \frac{1}{2}\left(\frac{1}{s+2}\right) + \frac{1}{3}\left(\frac{1}{s+3}\right)$$

ILT

$$\boxed{y(t) = \frac{1}{6} u(t) - \frac{1}{2} e^{-2t} u(t) + \frac{1}{3} e^{-3t} u(t)}$$

2Q, solve the differential Equation.

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = 2\cdot\frac{dx(t)}{dt} + 4x(t)$$

using L·T where $x(t) = e^{-2t}u(t)$

<u>Sol</u> Apply LT on b·s of diff equ

$$L\left[\frac{d^2y(t)}{dt}\right] + 4L\left[\frac{dy(t)}{dt}\right] + 3L[y(t)] = 2\cdot L\left[\frac{dx(t)}{dt}\right] + 4L[x(t)]$$

$$[s^2Y(s) - s\underbrace{y(0^-)}_{0} - \underbrace{y'(0^-)}_{0}] + 4[s\cdot Y(s) - \underbrace{y(0^-)}_{0}] + 3Y(s) =$$

$$2[sX(s) - \underbrace{x(0^-)}_{0}] + 4X(s)$$

$$Y(s)[s^2+4s+3] = X(s)[2s+4]$$

WKT $x(t) = e^{-2t}u(t) \xleftrightarrow{LT} X(s) = \frac{1}{s+2}$

$$\Rightarrow Y(s)[s^2+4s+3] = \frac{1}{s+2}[2s+4]$$

$$Y(s) = \frac{2}{s^2+4s+3} = \frac{2}{(s+1)(s+3)} = \frac{C_1}{s+1} + \frac{C_2}{s+3}$$

If $s=-1 \Rightarrow 2 = 2C_1 \Rightarrow C_1 = 1$

$s=-3 \Rightarrow \cancel{2} = -\cancel{2}C_2 \Rightarrow C_2 = -1$

$$Y(s) = \frac{1}{s+1} - \frac{1}{s+3}$$

ILT

$$\boxed{y(t) = e^{-t}u(t) - e^{-3t}u(t)}$$

3Q, A system is described by the following diff equ

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = \frac{d\,x(t)}{dt} + x(t)$$

where $y'(\bar{0}) = 3,\ y(\bar{0}) = 1,\ x(t) = u(t)$

sol

$$L\left[\frac{d^2y(t)}{dt^2}\right] + 6L\left[\frac{dy(t)}{dt}\right] + 8L[y(t)] = L\left[\frac{d\,x(t)}{dt}\right] + L[x(t)]$$

$$[s^2Y(s) - s\,y(0^-) - y'(\bar{0})] + 6[sY(s) - y(\bar{0})] + 8Y(s) =$$

$$[sX(s) - \underbrace{x(\bar{0})}_{0}] + X(s)$$

$$[s^2Y(s) - s - 3] + 6[sY(s) - 1] + 8Y(s) = \underbrace{X(s)}_{1/s}[s+1]$$

$$Y(s)[s^2+6s+8] - s - 3 - 6 = \frac{1}{s}[s+1]$$

$$Y(s)[s^2+6s+8] = s + 10 + \frac{1}{s}$$

$$Y(s) = \frac{s^2+10s+1}{s(s^2+6s+8)} = \frac{s^2+10s+1}{s(s+2)(s+4)} = \frac{c_1}{s} + \frac{c_2}{s+2} + \frac{c_3}{s+4}$$

If $s = 0 \Rightarrow 1 = 8c_1 \Rightarrow c_1 = 1/8$.

$s = -2 \Rightarrow -15 = -4c_2 \Rightarrow c_2 = 15/4$

$s = -4 \Rightarrow -23 = +8c_3 \Rightarrow c_3 = -23/8$

$$Y(s) = \frac{1}{8}\left(\frac{1}{s}\right) + \frac{15}{4}\left(\frac{1}{s+2}\right) - \frac{23}{8}\left(\frac{1}{s+4}\right)$$

ILT

$$\boxed{y(t) = \frac{1}{8}u(t) + \frac{15}{4}e^{-2t}u(t) - \frac{23}{8}e^{-4t}u(t)}$$

4Q, Find the initial value, Final value of following

$$X(s) = \frac{3s+2}{s(s^2+3s+2)}$$

sol: • By initial value theorem

$$x(0) = \lim_{s\to\infty} sX(s)$$

$$= \lim_{s\to\infty} \cancel{s}\,\frac{3s+2}{\cancel{s}(s^2+3s+2)}$$

$$= \lim_{s\to\infty} \frac{\cancel{s}(3+2/s)}{s^{\cancel{2}}\left(1+\frac{3}{s}+\frac{2}{s^2}\right)}$$

$$= 0$$

• By final value theorem

$$x(\infty) = \lim_{s\to 0} sX(s)$$

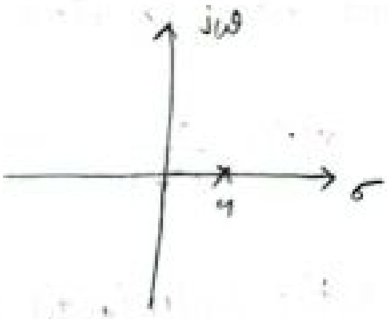
$$= \lim_{s\to 0} \frac{3s+2}{(s+1)(s+2)}$$

$$= \frac{0+2}{(1)(2)} = 1$$

5Q, Find the final value of the following

1) $X(s) = \frac{1}{s-4}$

The $sX(s) = \frac{s}{s-4}$

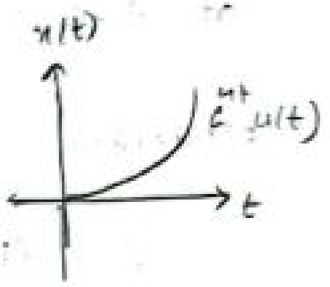


Poles $\Rightarrow s = 4$

The pole is located at RHS of s-plane.

$\therefore$ we cannot apply F.V.T

- $x(t) = e^{4t}u(t) \xleftrightarrow{LT} X(s) = \frac{1}{s-4}$



$x(t) = e^{4t}u(t)$

Final value is ∞

$x(\infty) = e^{\infty} = \infty$

2) $X(s) = \frac{s-2}{s(s+4)}$

Poles of $sX(s)$ are located at LHS of s-plane
$s = -4$

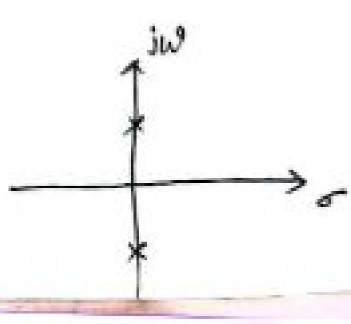
By FVT

$x(\infty) = \lim_{s \to 0} sX(s)$

$= \lim_{s \to 0} \cancel{s}\frac{s-2}{\cancel{s}(s+4)}$

$= \frac{-2}{4} = -1/2$

3) $X(s) = \frac{2}{s^2+3} \Rightarrow sX(s) = \frac{2s}{s^2+3}$



Poles :- $s^2+3 = 0$

$s = \pm j\sqrt{3}$

The poles of $sX(s)$ are located on the imag axis.

so, we need to find ILT

$$X(s) = \frac{2}{s^2+3}$$

$$x(t) = 2\sin(\sqrt{3}t) \xleftrightarrow{LT} X(s) = \frac{2(\sqrt{3})}{s^2+3}$$

$$x(t) = \frac{2}{\sqrt{3}}\sin(\sqrt{3}t)$$

at $t \to \infty$ the signal $x(t)$ is having oscillatory behaviour b/w -1 to $+1$

$\therefore$ final value can't determine.

4) $$X(s) = \frac{2}{s^2+4s-2}$$

$$s^2+4s-2 = 0 \Rightarrow s = \frac{-4 \pm \sqrt{16+8}}{2}$$

$$= \frac{-4 \pm \sqrt{24}}{2} = -2 \pm \sqrt{6}$$