⑪ Integration in s-domain property

If $x(t) \xleftrightarrow{LT} X(s)$: ROC: R

then $\frac{x(t)}{t} \xleftrightarrow{LT} \int_s^{\infty} X(s)ds$ ROC: R

Proof:-

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

Integrate w.r.t ds on Both sides

$$\int_s^{\infty} X(s)ds = \int_s^{\infty}\left[\int_{-\infty}^{\infty} x(t)e^{-st}dt\right]ds$$

$$\int_s^{\infty} X(s)ds = \int_{-\infty}^{\infty}\left[x(t)\int_s^{\infty} e^{-st}dt\right]dt$$

$$\Rightarrow \int_s^{\infty} e^{-st}dt = \left[\frac{e^{-st}}{-t}\right]_s^{\infty}$$

$$= \frac{-1}{t}\left[e^{-\infty t} - e^{st}\right]$$

$$= \frac{1}{t}e^{st}$$

$$\int_s^{\infty} X(s)ds = \int_{-\infty}^{\infty} \frac{x(t)}{t}e^{-st}dt$$

$$\boxed{\frac{x(t)}{t} \xleftrightarrow{LT} \int_s^{\infty} X(s)ds}$$

Ⓔ find the LT of signal $x(t) = \frac{\sin(\omega t)u(t)}{t}$

$$y(t) = \sin(\omega t) \xleftrightarrow{LT} Y(s) = \frac{\omega}{s^2+\omega^2} \quad \text{ROC: } \sigma>0$$

Apply integration in s-domain property

$$\frac{y(t)}{t} \xleftrightarrow{LT} \int_s^{\infty} Y(s)ds$$

$$L\left[\frac{\sin(\omega t)u(t)}{t}\right] = \int_s^{\infty}\frac{\omega}{s^2+\omega^2}ds$$

$$= \tan^{-1}\left(\frac{s}{\omega}\right)\Big|_s^{\infty}$$

$$= \tan^{-1}(\infty) - \tan^{-1}\left(\frac{s}{\omega}\right)$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{\omega}\right)$$

$$= \cot^{-1}\left(\frac{s}{\omega}\right)$$

Inverse laplace transform calculations:

→ calculate the ILF for following. X(s)

① $X(s) = \frac{2s+4}{s^2+4s+3}$

ⓐ ROC: $Re(s) > -1$ ⓑ $Re(s) < -3$ ⓒ $-3 < Re(s) < -1$

$$X(s) = \frac{2s+4}{(s+1)(s+3)} =$$

Two poles $-1, -3$ are real and distinct

∴ partial fractions are

$$X(s) = \frac{C_1}{s+1} + \frac{C_2}{s+3}$$

$C_1 = (s+1)X(s)\Big|_{s=-1} = \frac{(s+1)(2s+4)}{(s+1)(s+3)}\Big|_{s=-1} = \frac{2(-1)+4}{-1+3} = 1$

$C_2 = (s+3)X(s)\Big|_{s=-3} = \frac{(s+3)(2s+4)}{(s+1)(s+3)}\Big|_{s=-3} = 1$

$\therefore X(s) = \frac{1}{s+1} + \frac{1}{s+3}$

ⓐ $Re(s) > -1$

$\Rightarrow \sigma > -1$

∴ signal must be causal

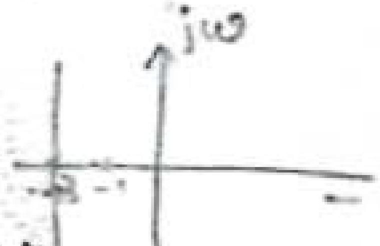
∴ causal solution for $\frac{1}{s+a}$ is $e^{-at}u(t)$

$\therefore x(t) = e^{-t}u(t) + e^{-3t}u(t)$

ⓑ $Re(s) < -3$

$\therefore \sigma < -3$

signal must be anticausal



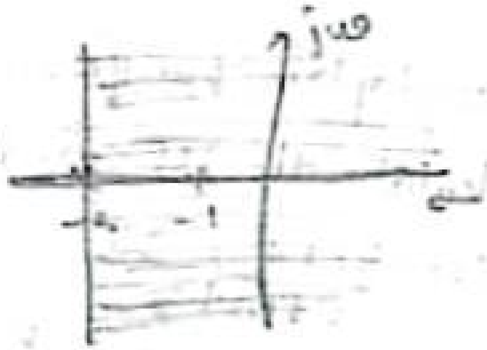
$x(t) = -e^{-t}u(-t) - e^{-3t}u(-t)$

ⓒ $-3 < Re(s) < -1$

∴ signal must be noncausal

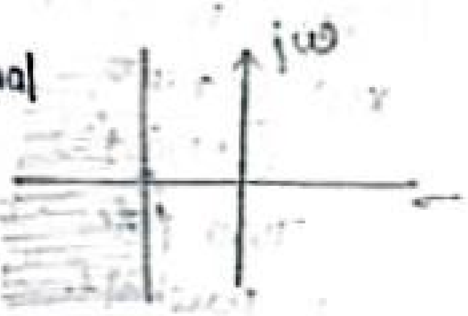
$Re(s) > -3$

∴ Pole corresponding to -3 is a causal signal



$Re(s) < -1$

∴ Pole corresponding to -1 is a anti-causal signal



② find the ILF of $X(s) = \frac{s^2+2s+5}{(s+3)(s+5)^2}$

ROC → $Re(s) > -3$

The poles are $-3, -5, -5$, the poles are real and repeated

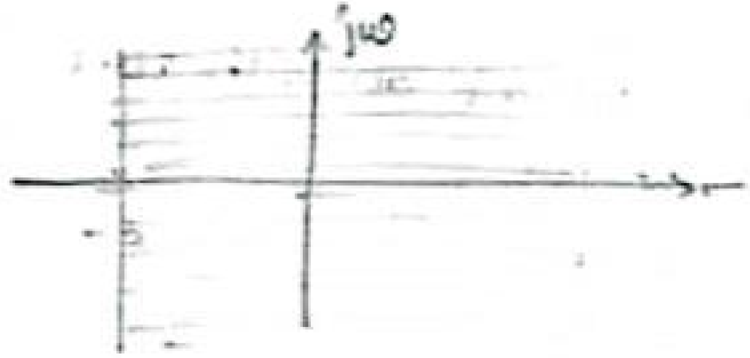
$X(s) = \frac{s^2+2s+5}{(s+3)(s+5)^2} = \frac{c_1}{s+3} + \frac{\lambda_1}{(s+5)} + \frac{\lambda_2}{(s+5)^2}$

$c_1 = (s+3)X(s)\Big|_{s=-3} = 2$

$\lambda_2 = (s+5)^2 X(s)\Big|_{s=-5} = -10$

$\lambda_1 = \frac{d}{ds}\left[(s+5)^2 X(s)\right]\Big|_{s=-5} = -1$

$X(s) = \frac{2}{s+3} - \frac{1}{s+5} - \frac{10}{(s+5)^2}$



∴ all signals are causal signals

$x(t) = 2e^{-3t}u(t) - e^{-5t}u(t) - 10te^{-5t}u(t)$

③ find the ILF of $X(s) = \frac{3s^2+7s+4}{s[(s+2)^2+9]}$ if ROC is no mention choose causal

The poles are $0, -2 \pm 3j$. ∴ poles are complex conjugate

$$X(s) = \frac{A}{s} + \frac{Bs+C}{(s+2)^2+9}$$

$$\frac{3s^2+7s+4}{s[(s+2)^2+9]} = \frac{A[(s+2)^2+9]+(Bs+C)s}{s[(s+2)^2+9]}$$

$$3s^2+7s+4 = A[(s+2)^2+9]+(Bs+C)s$$

put $s=0$

$$4 = A[4+9]+0$$

$$A = \frac{4}{13}$$

compare s^2 coefficient on both side

$$3 = A+B$$

$$3 = \frac{4}{13}+B$$

$$B = 3-\frac{4}{13}$$

$$B = \frac{35}{13}$$

compare s coefficient on both side

$$7 = 4A + C$$

$$C = 7-4A$$

$$= 7-\frac{16}{13}$$

$$C = \frac{75}{13}$$

$$X(s) = \frac{4}{13s} + \frac{(35/13)s + (75/13)}{(s+2)^2+3^2}$$

$$X(s) = \frac{4}{13}\cdot\frac{1}{s} + \frac{\frac{35}{13}(s+2-2)}{(s+2)^2+3^2} + \frac{75}{13}\left[\frac{1}{(s+2)^2+3^2}\right]$$

$$= \frac{4}{13}\cdot\frac{1}{s} + \frac{35}{13}\left[\frac{s+2}{(s+2)^2+3^2}\right] + \frac{1}{(s+2)^2+3^2}\left[\frac{75}{13}-\frac{70}{13}\right]$$

$$x(t) = \frac{4}{13}\left[\frac{1}{s}\right] + \frac{35}{13}\left[\frac{s+2}{(s+2)^2+3^2}\right] + \frac{5}{13\times3}\left[\frac{3}{(s+2)^2+3^2}\right]$$

ILT

$$x(t) = \frac{4}{13}u(t) + \frac{35}{13}e^{-2t}\cos(3t)u(t) + \frac{5}{39}e^{-2t}\sin(3t)u(t)$$

④ find the ILT of the $X(s) = \frac{s^3+1}{s(s+1)(s+2)}$

$$X(s) = \frac{s^3+1}{s^3+3s^2+2s}$$

By using long division method

$$s^3+3s^2+2s \;)\; s^3+1 \;(\; 1$$

$$\underline{s^3+3s^2+2s}$$

$$1-3s^2-2s$$

$$\therefore \frac{s^3+1}{s^3+3s^2+2s} = 1 + \frac{1-3s^2-2s}{s^3+3s^2+2s}$$

$$X(s) = 1 - \left[\frac{3s^2+2s-1}{s(s+1)(s+2)}\right]$$

$$\therefore X(s) = 1 - X_1(s)$$

where

$$X_1(s) = \frac{3s^2+2s-1}{s(s+1)(s+2)}$$

$$\therefore X_1(s) = \frac{C_1}{s} + \frac{C_2}{s+1} + \frac{C_3}{s+2}$$

$$C_1 = sX_1(s)\big|_{s=0} = -1/2$$

$$C_2 = (s+1)X_1(s)\big|_{s=-1} = 0$$

$$C_3 = (s+2)X_1(s)\big|_{s=-2} = 7/2$$

$$\therefore X_1(s) = \frac{-1}{2s} + 0 + \frac{7}{2}\cdot\frac{1}{(s+2)}$$

ILT

$$x(t) = \delta(t) + \frac{1}{2}u(t) - \frac{1}{2}e^{-2t}u(t)$$

(5) find the ILT of $X(s) = \frac{1+e^{-2s}}{3s^2+2s}$

let $X_1(s) = \frac{1}{3s^2+2s}$

$X(s) = (1+e^{-2s})X_1(s)$

$\because X_1(s) = \frac{1}{s(3s+2)} = \frac{C_1}{s} + \frac{C_2}{3s+2}$

$C_1 = sX(s)\Big|_{s=0} \Rightarrow \frac{s\cdot 1}{s(3s+2)} = 1/2$

$C_2 = (3s+2)X(s)\Big|_{s=-\frac{2}{3}} \Rightarrow \frac{1}{s} = -\frac{3}{2}$

$\because X_1(s) = \frac{1}{2}\left(\frac{1}{s}\right) - \frac{3}{2}\left(\frac{1}{3s+2}\right)$

$X_1(s) = \frac{1}{2}\left(\frac{1}{s}\right) - \frac{1}{2}\left(\frac{1}{s+\frac{2}{3}}\right)$

$\therefore X(s) = (1+e^{-2s})X_1(s)$

$= X_1(s) + e^{-2s}X_1(s)$

ILT

$x(t) = x_1(t) + x_1(t-2)$

$x(t) = \frac{1}{2}u(t) - \frac{1}{2}e^{-2/3}u(t)$

$x(t) = \frac{1}{2}u(t) - \frac{1}{2}e^{-2/3}u(t) + x_1(t-2)$

(6) find the ILT of the following $X(s)$ using convolution theorem $X(s) = \frac{1}{(s+1)(s^2+1)}$

$X_1(s) = \frac{1}{s+1}$

$X_2(s) = \frac{1}{s^2+1}$

Apply ILT on Both sides

$L^{-1}[X(s)] = L^{-1}[X_1(s)X_2(s)]$

$x(t) = x_1(t) * x_2(t)$

$x_1(t) = e^{-t}u(t) \xleftrightarrow{LT} X_1(s) = \frac{1}{s+1}$

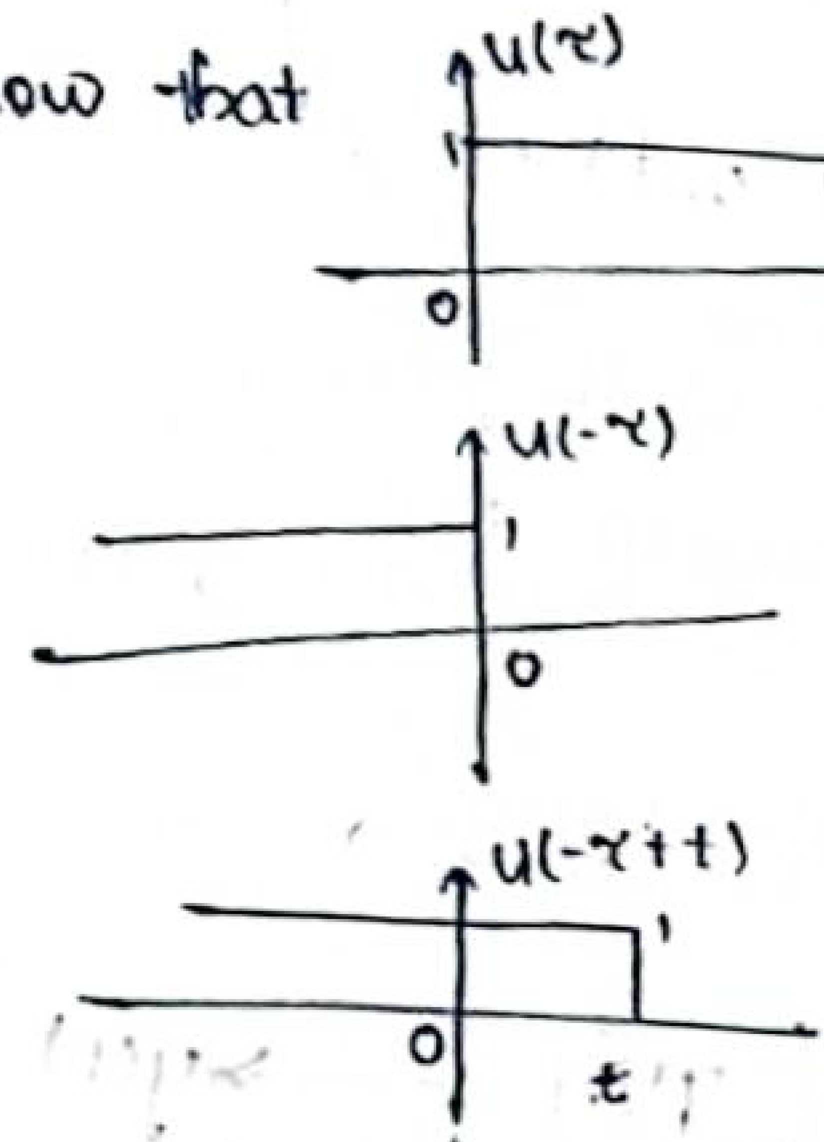
$x_2(t) = \sin(t)u(t) \xleftrightarrow{LT} X_2(s) = \frac{1}{s^2+1}$

$\because x(t) = x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)\,d\tau$

$= \int_{-\infty}^{\infty} x_2(\tau)x_1(t-\tau)\,d\tau$

$\therefore \int_{-\infty}^{\infty} \sin(\tau)u(\tau)\,e^{-(t-\tau)}u(t-\tau)\,d\tau$

we know that



$\therefore x(t) = \int_0^t e^{-(t-\tau)}\sin(\tau)\,d\tau$

$= \int^t e^{-t}\cdot e^{\tau}\sin(\tau)\,d\tau$

$$x(t) = e^{-t}\int_0^t e^{\tau}\sin(\tau)\,d\tau$$

$\int e^{ax}\sin(bx)\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin(bx) - b\cos(bx)]$

$$x(t) = e^{-t}\left[\frac{e^{\tau}}{2}\left[\sin(\tau) - \cos(\tau)\right]\right]_0^t$$

(or) $\int uv = uv_1 - \int v_1 u'$, $v_1 = \int v$, $u' = u$

$$\cancel{x(t) = \frac{e^{-t}}{2}\left[(e^{t} - e^{0})[\sin t - \sin 0 - \cos t + \cos 0]\right]}$$

$$\cancel{x(t) = \frac{e^{-t}}{2}\left[(e^{t}-1)[\sin t - 2\cos t + 1]\right]}$$

$$x(t) = e^{-t}\left[\frac{e^t}{2}(\sin t - \cos t) - \frac{e^0}{2}(\sin 0 - \cos 0)\right]$$

$$= e^{-t}\left[\frac{e^t}{2}(\sin t - \cos t) - \frac{1}{2}(0-1)\right]$$

$$= e^{-t}\left[\frac{e^t}{2}(\sin t - \cos t) + \frac{1}{2}\right]$$

$$x(t) = \frac{1}{2}(\sin t - \cos t) + \frac{1}{2}e^{-t}$$

Q) find the ILT of the following $X(s)$ using convolution theorem $X(s) = \frac{1}{s(s^2+1)}$

Poles and zeros of a system:

$$x(t) \longrightarrow \boxed{T[\cdot]} \longrightarrow y(t)$$

LTI system
(continous)

for any LTI system, the system can always be represented by using integro differential eq^n

$$y(t) = -\sum_{k=1}^{N} a_k \frac{d^k y(t)}{dt^k} + \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

where a_k and b_k are constants.

- Apply LT on both sides

$$Y(s) = -\sum_{k=1}^{N} a_k s^k Y(s) + \sum_{k=0}^{M} b_k s^k X(s)$$

$$Y(s) + \sum_{k=1}^{N} a_k s^k Y(s) = \sum_{k=0}^{M} b_k s^k X(s)$$

$$Y(s)\left[1 + \sum_{k=1}^{N} a_k s^k\right] = \left[\sum_{k=0}^{M} b_k s^k\right] X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{1 + \sum_{k=1}^{N} a_k s^k}$$

$H(s)$ is called transfer function of the system.
where $N \geq M$
where N is called order of the system

$$\boxed{H(s) = \frac{b_0 + b_1 s + b_2 s^2 + b_3 s^3 + \cdots + b_M s^M}{1 + a_1 s + a_2 s^2 + a_3 s^3 + \cdots + a_N s^N}}$$

zeros of the system can be computed by equating Numerating polynomial of $H(s) = 0$
zeros are always denoted by circule symbol in s-plane at zero location the value of the transfer function $H(s) = 0$

polynomial of $H(s)=0$

poles are denote by x symbol in s-plane.

at pole locations the value of $H(s)$ is ∞

Impulse response and transfer function of system

$$x(t) \xrightarrow{i/p} \boxed{h(t)} \xrightarrow{o/p} y(t) = x(t) * h(t)$$

LTI system

$$\boxed{y(t) = x(t) * h(t)}$$

apply L.T on both sides

$$L[y(t)] = L[x(t) * h(t)]$$

apply convolution theorm

$$L[y(t)] = L[x(t)]\,L[h(t)]$$

$$\boxed{Y(s) = X(s)\,H(s)} \rightarrow \text{o/p of LTI system}$$

$$\therefore y(t) = L^{-1}[Y(s)]$$

$H(s) = \dfrac{Y(s)}{X(s)}$ → Transfer function (or) system function

$$H(s) = L[h(t)]$$

$$\boxed{h(t) \xleftrightarrow{LT} H(s)}$$

where $h(t)$ → impulse response of the system.

$H(s)$ → Transfer function of the system

① Input to the LTI system is $x(t) = e^{-2t}u(t)$ and the impulse response of the system is $h(t) = e^{-3t}u(t)$ find the output of the system

$x(t)$

$$X(s) \rightarrow \boxed{H(s)} \rightarrow Y(s) = X(s)H(s)$$

$$\therefore Y(s) = X(s)H(s)$$

$$x(t) = e^{-2t}u(t) \xleftrightarrow{LT} X(s) = \frac{1}{s+2}$$

$$h(t) = e^{-3t}u(t) \xrightarrow{LT} H(s) = \frac{1}{s+3}$$

$$\therefore Y(s) = X(s)H(s) = \frac{1}{(s+2)(s+3)}$$

using partial fractions

$$Y(s) = \frac{1}{s+2} - \frac{1}{s+3}$$

$$y(t) = e^{-2t}u(t) - e^{-3t}u(t)$$

② The o/p of the systems is $y(t) = 2e^{-3t}u(t)$ when the input $x(t) = u(t)$

① find the impluse response of the system

② find the o/p of the system when the input is $e^{-t}u(t)$

$$y(t) = 2e^{-3t}u(t), \; x(t) = u(t)$$

$$x(t) = u(t) \xrightarrow{i/p} \boxed{h(t)} \xrightarrow{o/p} y(t) = 2e^{-3t}u(t)$$

$$Y(s) = X(s)H(s)$$

$$H(s) = \frac{Y(s)}{X(s)}$$

$$x(t) = u(t) \xrightarrow{LT} X(s) = \frac{1}{s}$$

$$y(t) = 2e^{-3t}u(t) \xrightarrow{LT} X(s) = \frac{2}{s+3}$$

$H(s) = \frac{(s+3)}{\frac{1}{s}}$

$H(s) = \frac{2s}{s+3} = 2\frac{[s+3-3]}{s+3} = 2\left[1 - \frac{3}{s+3}\right]$

$h(t) = \mathcal{L}^{-1}[H(s)] = 2\left[\delta(t) - 3e^{-3t}u(t)\right]$

step response of the system:

$x(t) \xrightarrow{i/p} \boxed{h(t)} \xrightarrow{o/p} y(t) = s(t)$

$= u(t)$

step response of the system means whose i/p is $u(t)$

o/p of the system is $Y(s) = X(s)H(s)$

$x(t) = u(t) \xleftrightarrow{LT} X(s) = \frac{1}{s}$

$Y(s) = \frac{H(s)}{s}$

$y(t) = \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[\frac{H(s)}{s}\right] = \int_0^t h(t)\,dt$

$y(t) = s(t) \rightarrow$ step response of the system

$$\boxed{s(t) = \int_{-\infty}^{t} h(t)\,dt}$$

$$\boxed{h(t) = \frac{ds(t)}{dt}}$$

(Q) The step response of the system is $e^{-3t}u(t)$ and then find the o/p of the system for the input $e^{-2t}u(t)$

$s(t) = e^{-3t}u(t)$

$h(t) = \frac{d}{dt}(e^{-3t}u(t))$

$= (-3)e^{-3t}u(t) + e^{-3t}\delta(t)$

$e^{-3t}\delta(t)$

$h(t) = -3e^{-3t}u(t) + \delta(t)$

$x(t)$

$h(t) = -3e^{-3t}u(t) + \delta(t) \xleftrightarrow{LT} H(s) = \frac{-3}{s+3} + 1$

$= \frac{-3+s+3}{s+3}$

$= \frac{s}{s+3}$

$Y(s) = X(s)H(s)$

$= \frac{s}{(s+2)(s+3)} = \frac{C_1}{(s+2)} + \frac{C_2}{(s+3)}$

$C_1 = (s+2)Y(s)\big|_{s=-2} = \frac{s}{s+3} = \frac{-2}{1} = -2$

$C_2 = (s+3)Y(s)\big|_{s=-3} = \frac{s}{s+2} = \frac{-3}{-1} = +3$

$\therefore Y(s) = \frac{-2}{s+2} + \frac{3}{s+3}$

$y(t) = -2e^{-2t}u(t) + 3e^{-3t}u(t)$

causal system

Impulse response of the system $h(t)$ is '0' for $t<0$ then the system is causal system

$h(t) \xleftrightarrow{TF} H(s)$

IR TF

ROC: of $H(s)$ is $\longrightarrow$ < man

stable system:-

If $\int_{-\infty}^{\infty} |h(t)|\,dt < \infty$

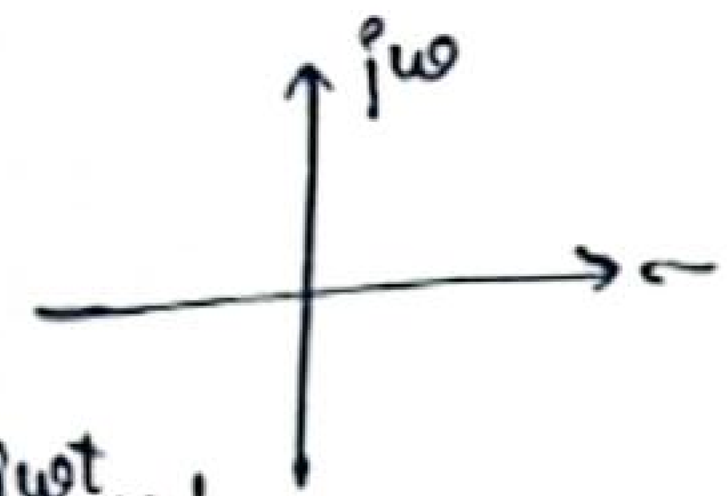
Then the system is stable system

Transfer function

$H(s) = \mathcal{L}[h(t)] = \int_{-\infty}^{\infty} h(t)e^{-st}\,dt$

Assume $\sigma = 0$

$s = j\omega$



$$\therefore H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t}\, dt$$

$$|H(j\omega)| = \left| \int_{-\infty}^{\infty} h(t) e^{-j\omega t}\, dt \right|$$

$$\leq \int_{-\infty}^{\infty} |h(t)|\,|e^{-j\omega t}|\, dt$$

$$|H(j\omega)| \leq \int_{-\infty}^{\infty} |h(t)|\, dt$$

for stable system

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

$$|H(j\omega)| < \infty$$

$$H(s)\big|_{s=j\omega} < \infty$$

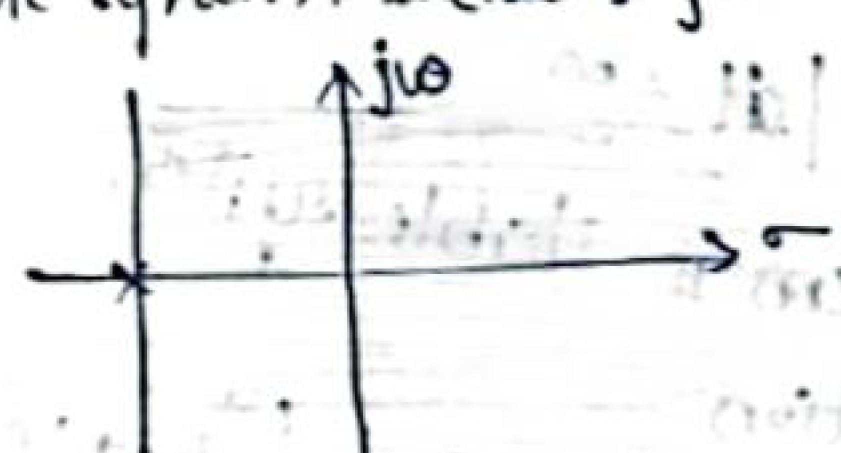
ROC of $H(s)$ must include the $j\omega$ axis for stable system.

⊛ causal and stable system

ROC of $H(s)$

causal system : $\sigma > \sigma_{max}$

stable system : Includes $j\omega$ axis



The all poles of causal &

lies on the LHS of s-plane

For causal & stable system all the poles of $H(s)$ must lies on the LHS of s-plane.

unilateral laplace transform

This is also know as one sided laplace transform

$$x(t) \xleftrightarrow{LT} X(s)$$

Bilateral laplace transform (BLT)

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st}\, dt$$

ULT

$$X(s) = \int_{0}^{\infty} x(t) e^{-st}\, dt$$

If the signal is causal signal i.e, $x(t) = 0$ for $t < 0$

$$\therefore BLT \Rightarrow X(s) = \int_{0}^{\infty} x(t) e^{-st}\, dt$$

for causal signal both ULT & BLT is same and its ROC $\sigma > \sigma_{max}$

① Differentation propertie

If $x(t) \xleftrightarrow{LT} X(s)$

then $\dfrac{dx(t)}{dt} \xleftrightarrow{LT} sX(s) - x(0)$

Proof:

$$LT\left[\frac{dx(t)}{dt}\right] = \int_{0}^{\infty} \left[\frac{dx(t)}{dt}\right] e^{-st}\, dt$$

$$= \left[e^{-st} x(t)\right]_{0}^{\infty} - \int_{0}^{\infty} x(t) e^{-st}(-s)\, dt$$

$$= \left[e^{-\infty}x(\infty) - e^{0}x(0)\right] + s\int_0 x(t)\, e^{\cdot}\, dt$$

$$= 0 - x(0) + sX(s)$$

$$= sX(s) - x(0)$$

$$\therefore \boxed{\frac{dx(t)}{dt} \xleftrightarrow{LT} sX(s) - x(0)}$$

$$L\left[\frac{dx(t)}{dt}\right] = sX(s) - x(0)$$

$$L\left[\frac{dx(t)}{dt}\right] = sL[x(t)] - x(0)$$

similarly

$$L\left[\frac{d^2x(t)}{dt^2}\right] = L\left[\frac{d}{dt}\left(\frac{dx(t)}{dt}\right)\right] = sL\left[\frac{dx(t)}{dt}\right] - x'(0)$$

$$= s[sX(s) - x(0)] - x'(0)$$

$$\boxed{L\left[\frac{d^2x(t)}{dt^2}\right] = s^2X(s) - sx(0) - x'(0)}$$

$$\boxed{L\left[\frac{d^3x(t)}{dt^3}\right] = s^3X(s) - s^2x(0) - sx'(0) - x''(0)}$$

② Integration propertiy

If $x(t) \xleftrightarrow{LT} X(s)$

then $\int_{-\infty}^{t} x(t)\, dt \xleftrightarrow{LT} \frac{X(s)}{s} + \int_{-\infty}^{t} \frac{x(0)}{s}\, dt$

proof:- $f(t) = \int_{-\infty}^{t} x(t)\, dt$

$$\frac{df(t)}{dt} = x(t)$$

$$L\left[\frac{df(t)}{dt}\right] = L[x(t)]$$

$$sF(s) - f(\bar{0}) = X(s)$$

$$sF(s) = X(s) + f(\bar{0})$$

$$F(s) = \frac{X(s)}{s} + \frac{1}{s} f(\bar{0})$$

$$f(\bar{0}) = \int_{-\infty}^{0} x(\tau)\, d\tau$$

$$F(s) = \frac{X(s)}{s} + \frac{1}{s}\int_{-\infty}^{0} x(0)\, dt$$

$\downarrow$

$L[f(t)]$

③ Initial value theorem

If $x(t) \xleftrightarrow{LT} X(s)$

then $x(0) = \lim_{s\to\infty} sX(s)$

proof:-

$$L\left[\frac{dx(t)}{dt}\right] = \int_0^\infty \frac{dx(t)}{dt} e^{-st}\, dt$$

$$sX(s) - x(0) = \int_0^\infty \frac{dx(t)}{dt} e^{-st}\, dt$$

$$\lim_{s\to\infty}[sX(s) - x(0)] = \lim_{s\to\infty}\left[\int_0^\infty \frac{dx(t)}{dt} e^{-st}\, dt\right]$$

$$\lim_{s\to\infty} sX(s) - x(0) = 0$$

$$\boxed{x(0) = \lim_{s\to\infty} sX(s)}$$

If $x(t) \xleftrightarrow{LT} X(s)$

then $x(\infty) = \lim_{s \to 0} sX(s)$

proof: $L\left[\frac{dx(t)}{dt}\right] = \int_0^{\infty} x(t) e^{-st} dt$

$sX(s) - x(0) = \int_0^{\infty} \frac{dx(t)}{dt} e^{-st} dt$

$\lim_{s \to 0} [sX(s) - x(0)] = \lim_{s \to 0} \left[\int_0^{\infty} \frac{dx(t)}{dt} e^{-st} dt \right]$

$\lim_{s \to 0} sX(s) - x(0) = \int_0^{\infty} \frac{dx(t)}{dt} \left(\lim_{s \to 0} e^{-st} \right) dt$

$\therefore \lim_{s \to 0} sX(s) - x(0) = x(\infty) - x(0)$

$x(\infty) = \lim_{s \to 0} sX(s)$

This theorem is applicable if & if only all the poles of $s \cdot X(s)$ must lies on the LHS of s-plane

* Q- A system is discribe by the following differential equation $\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = x(t)$

(a) find the implue respones of the system

(b) find the step respones of the system

(a) $x(t) = \delta(t)$

Given $\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = x(t)$

apply LT on Both sides

$L\left[\frac{d^2y \ldots}{dt}\right] \quad \left[\frac{}{dt}\right]$

The initial condition values are not given

$\therefore$ all initial condn values are zeros

$[s^2Y(s) - s^1y(0^-) - y'(0^-)] + 5[sY(s) - y(0^-)] + 6Y(s) = X(s)$

$s^2Y(s) + 5sY(s) + 6Y(s) = X(s)$

$Y(s)[s^2 + 5s + 6] = X(s)$

$x(t) = \delta(t) \xleftrightarrow{LT} X(s) = 1$

$\therefore Y(s) = \frac{1}{s^2 + 5s + 6} = \frac{1}{(s+2)(s+3)} = \frac{1}{s+2} - \frac{1}{s+3}$

Apply ILT

$y(t) = e^{-2t}u(t) - e^{-3t}u(t)$

(b) $x(t) = u(t)$

given $\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = x(t)$

apply LT on Both side

$\therefore s^2Y(s) + 5sY(s) + 6Y(s) = X(s)$

$Y(s)[s^2 + 5s + 6] = X(s)$

$x(t) = u(t) \xleftrightarrow{LT} X(s) = \frac{1}{s}$

$\therefore Y(s) = \frac{1}{s(s^2 + 5s + 6)} = \frac{1}{s(s+2)(s+3)} = \frac{C_1}{s} + \frac{C_2}{s+2} + \frac{C_3}{s+3}$

$\therefore Y(s) = \frac{1}{6s} + \left(\frac{-1}{2}\right)\frac{1}{s+2} + \frac{1}{3}\left(\frac{1}{s+3}\right)$

Apply ILT

$y(t) = \frac{1}{6}u(t) - \frac{1}{2}e^{-2t}u(t) + \frac{1}{3}e^{-3t}u(t)$

⑤ solve the following differential eqn using L.T approach $\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = 2\frac{dx(t)}{dt} + 4x(t)$

where $x(t) = e^{-2t}u(t)$

Given $\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = 2\frac{dx(t)}{dt} + 4x(t)$

apply LT on Both side

$$L\left[\frac{d^2y(t)}{dt^2}\right] + 4L\left[\frac{dy(t)}{dt}\right] + 3L[y(t)] = 2L\left[\frac{dx(t)}{dt}\right] + 4L$$

$$[s^2Y(s) - s'Y(\bar{0}) - Y(0)] + 4[sY(s) - Y(0)] + 3Y(s)$$
$$= 2[sX(s) - X(0)] + 4X(s)$$

$$\Rightarrow s^2Y(s) + 4sY(s) + 3Y(s) = 2sX(s) + 4X(s)$$

$$Y(s)[s^2 + 4s + 3] = X(s)[2s + 4]$$

$$x(t) = e^{-2t}u(t) \xleftrightarrow{LT} \frac{1}{s+2}$$

$$Y(s)[s^2 + 4s + 3] = \frac{1}{(s+2)}[2s+4]$$

$$Y(s) = \frac{2}{s^2+4s+3}$$

$$Y(s) = \frac{2}{(s+1)(s+3)} = \frac{C_1}{(s+1)} + \frac{C_2}{(s+3)}$$

$$Y(s) = \frac{+1}{s+1} - \frac{1}{s+3}$$

apply ILT

$$\therefore y(t) = e^{-t}u(t) - e^{-3t}u(t)$$

⑥ A system is discribe by the following diff eqn

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = \frac{dx(t)}{dt} + x(t)$$

where $y'(\bar{0}) = 3$, $y(\bar{0}) = 1$ & $x(t) = u(t)$

Apply LT

$$[s^2Y(s) - sy'(\bar{0}) - Y(\bar{0})] + 6[sY(s) - Y(\bar{0})] + 8Y(s)$$
$$= (sX(s) - X(0)) + X(s)$$

$$[s^2Y(s) - s(1) - 3] + 6[sY(s) - 1] + 8Y(s)$$
$$= sX(s) + X(s)$$

$$Y(s)[s^2 + 6s + 8] - 9 - s = X(s)[s+1]$$

$$x(t) = u(t) \xleftrightarrow{LT} X(s) = \frac{1}{s}$$

$$Y(s)[s^2+6s+8] - 9 - s = \frac{1}{s}(s+1)$$

$$\therefore \cancel{Y(s) = \frac{s+1}{s(s^2+6s+8)} + \frac{9}{(s^2+6s+8)}}$$

$$Y(s)(s^2+6s+8) = 1 + \frac{1}{s} + 9 + s$$

$$Y(s)(s^2+6s+8) = 10 + s + \frac{1}{s}$$

$$= \frac{s^2+10s+1}{s}$$

$$Y(s) = \frac{s^2+10s+1}{s(s^2+6s+8)}$$

$C_1 = \frac{1}{8}$, $C_2 = \frac{15}{4}$

$C_3 = \frac{-23}{8}$

$$\therefore Y(s) = \frac{1}{8}\left(\frac{1}{s}\right) + \frac{15}{4}\left(\frac{1}{s+2}\right) - \frac{23}{8}\left(\frac{1}{s+4}\right)$$

ILT

$$\therefore y(t) = \frac{1}{8}u(t) + \frac{15}{4}e^{-2t}u(t) - \frac{23}{8}e^{-4t}u(t)$$

$$X(s) = \frac{3s+2}{s(s^2+3s+2)}$$ also find final value

From initial value theorem

$$x(0) = \lim_{s\to\infty} sX(s)$$

$$= \lim_{s\to\infty} \cancel{s}\left[\frac{3s+2}{\cancel{s}(s^2+3s+2)}\right]$$

$$= \lim_{s\to\infty} \frac{\cancel{s}(3+2/s)}{\cancel{s}(1+\frac{3}{s}+2/s^2)}$$

$\therefore x(0) = 0$

from final value theorem

$$\because X(s) = \frac{3s+2}{s(s^2+3s+2)} = \frac{3s+2}{s(s+1)(s+2)}$$

$$\therefore x(\infty) = \lim_{s\to 0} sX(s)$$

$$= \lim_{s\to 0} \cancel{s}\left[\frac{3s+2}{\cancel{s}(s^2+3s+2)}\right]$$

$$= \lim_{s\to 0} \frac{3s+2}{(s+1)(s+2)}$$

$$= \frac{0+2}{(0+1)(0+2)}$$

$\therefore x(\infty) = 1$

Q. find the final value for following $X(s)$

① $X(s) = \frac{1}{s-4}$ ④ $X(s) = \frac{2}{s^2+4s-2}$

② $X(s) = \frac{s-2}{s(s+4)}$

③ $X(s) = \frac{2}{s^2+3}$

①

pole $\Rightarrow s = 4$

$\therefore$ pole is located at RHS of s-plane

$\therefore$ we can't apply final value theorem

$\therefore$ apply ILT

$$x(t) = e^{4t}u(t) \xleftrightarrow{LT} X(s) = \frac{1}{s-4}$$

$$x(t) = e^{4t}u(t)$$

$$x(\infty) = e^{\infty} = \infty$$

② $X(s) = \frac{s-2}{s(s+4)}$

poles are located at LHS of s-plane

$\therefore$ from final value theorem

$$x(\infty) = \lim_{s\to 0} sX(s)$$

$$= \lim_{s\to 0} \cancel{s}\left(\frac{s-2}{\cancel{s}(s+4)}\right)$$

$$= \lim_{s\to 0} \frac{s-2}{s+4}$$

$$= \frac{0-2}{0+4}$$

$$= -\frac{1}{2}$$

③ $sX(s) = \frac{2s}{s^2+3}$

poles are $s^2+3=0$

$s = \pm j\sqrt{3}$

The poles of $sX(s)$ are located on the imaginary axis $\therefore$ we can't apply final value theorem

$$\therefore x(t) = 2\sin(\sqrt{3}t) \xleftrightarrow{LT} X(s) = \frac{2(\sqrt{3})}{s^2+3}$$

$x(t) = \frac{\ }{\sqrt{3}} \sin(\sqrt{3}t)$

at $t \to \infty$ the signal $x(t)$ is having osicallory behaviour b/w -1 to +1

∴ we can't apply final value can't determine

④ The poles of $sX(s)$ are: $-2 \pm j\sqrt{6}$

put $a = $

$-u(-n-1) \xleftrightarrow{ZT} \frac{1}{1-z^{-1}}$ ROC: $|z|<1$

Properties

① Linearity property:

If $x_1(n) \xleftrightarrow{ZT} X_1(z)$ ROC: r_1

$x_2(n) \xleftrightarrow{ZT} X_2(z)$ ROC: r_2

Then

$\alpha x_1(n) + \beta x_2(n) \xleftrightarrow{ZT} \alpha X_1(z) + \beta(X_2(z))$

ROC: $r_1 \cap r_2$

Proof:

$$Z[\alpha x_1(n) + \beta x_2(n)] = \sum_{n=-\infty}^{\infty} [\alpha x_1(n) + \beta x_2(n)] z^{-n}$$

$$= \alpha \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} + \beta \sum_{n=-\infty}^{\infty} x_2(n) z^{-n}$$

$$Z[\alpha x_1(n) + \beta x_2(n)] = \alpha X_1(z) + \beta X_2(z)$$

$$\alpha x_1(n) + \beta x_2(n) \xleftrightarrow{ZT} \alpha X_1(z) + \beta X_2(z)$$

$$Z[\alpha x_1(n) + \beta x_2(n)] = \alpha Z[x_1(n)] + \beta Z[x_2(n)]$$

Ⓠ find the z-transform of the signal $x(n) = a^n u(n) + b^n u(-n-1)$

① $b>a$ ② $b<a$

$$Z[x(n)] = Z[a^n u(n)] + Z[b^n u(-n-1)]$$

$$X(z) = \frac{1}{1-az^{-1}} - \frac{1}{1-bz^{-1}}$$

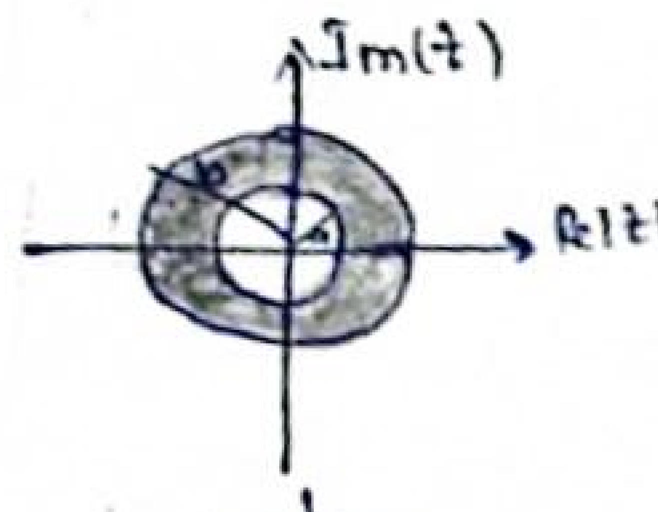
↓ROC ↓ROC

$|z|>|a|$ $|z|<|b|$

ⓐ $b>a$

ROC is angular region

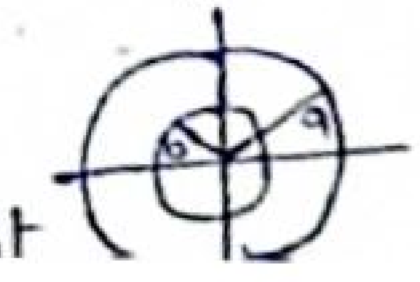
$|a|<|z|<|b|$



ⓑ $b<a$

Their is no common ROC

∴ z-Transform doesn't exist

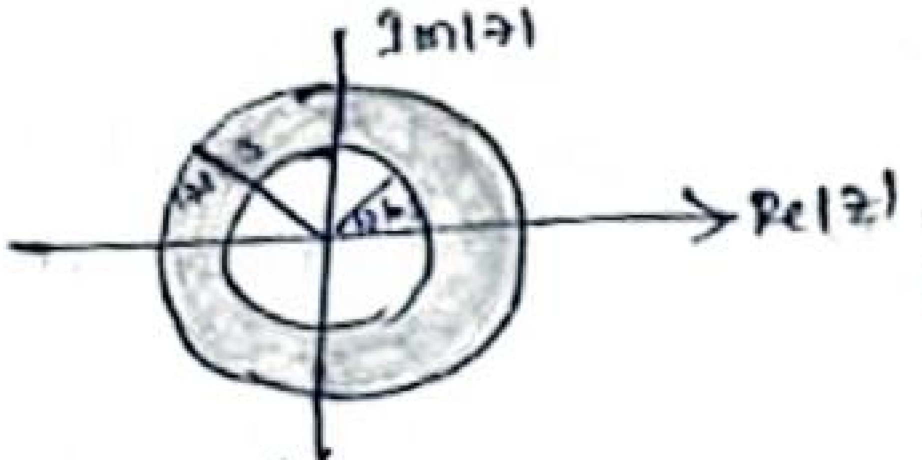


⑤ find z transform of $x(n) = 3(2)^n u(n) - 4(3)^n u(-n-1)$

$$z[x(n)] = z[3(2^n)u(n)] - z[4(3)^n u(-n-1)]$$

$$X(z) = \frac{3}{1-2z^{-1}} + \frac{4}{1-3z^{-1}}$$

$|z|>2 \qquad |z|<3$



⑥ find the z-Transform of

① $x(n) = \cos(\omega_0 n)\, u(n)$

② $x(n) = \sin(\omega_0 n)\, u(n)$

① we know that $\cos(\omega_0 n) = \frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2}$

$$\therefore x(n) = \left(\frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2}\right) u(n)$$

$$x(n) = \frac{1}{2}(e^{j\omega_0})^n u(n) + \frac{1}{2}(e^{-j\omega_0})^n u(n)$$

$$z[x(n)] = \frac{1}{2}\left[\frac{1}{1-e^{j\omega_0}z^{-1}}\right] + \frac{1}{2}\left[\frac{1}{1-e^{-j\omega_0}z^{-1}}\right]$$

↓ ROC $\qquad$ ↓ ROC

$|z| > |e^{j\omega_0}| \qquad |z| > |e^{-j\omega_0}|$

$$X(z) = \frac{1}{2}\left[\frac{1}{1-e^{j\omega_0}z^{-1}} + \frac{1}{1-e^{-j\omega_0}z^{-1}}\right]$$

$$= \frac{1}{2}\left[\frac{1-e^{-j\omega_0}z^{-1} + 1 - e^{j\omega_0}z^{-1}}{(1-e^{j\omega_0}z^{-1})(1-e^{-j\omega_0}z^{-1})}\right]$$

$$= \frac{1}{2}\left[\frac{2-e^{j\omega_0}z^{-1} - e^{-j\omega_0}z^{-1}}{(1-e^{j\omega_0}z^{-1})(1-e^{-j\omega_0}z^{-1})}\right]$$

$$\boxed{\quad 1-2z^{-1}\cos(\omega_0) + \dots}$$

② $x(n) = \sin(\omega_0 n)\, u(n)$

$$x(n) = \left(\frac{e^{j\omega_0 n} - e^{-j\omega_0 n}}{2j}\right) u(n)$$

$$x(n) = \frac{1}{2j} e^{j\omega_0 n} u(n) - \frac{1}{2j} e^{-j\omega_0 n} u(n)$$

$$z[x(n)] = \frac{1}{2j}\left[\frac{1}{1-e^{j\omega_0}z^{-1}}\right] - \frac{1}{2j}\left[\frac{1}{1-e^{-j\omega_0}z^{-1}}\right]$$

$$\therefore X(z) = \frac{1}{2j}\left[\frac{1}{1-e^{j\omega_0}z^{-1}} - \frac{1}{1-e^{-j\omega_0}z^{-1}}\right]$$

$$\boxed{X(z) = \frac{z^{-1}\sin(\omega_0)}{1-2z^{-1}\cos(\omega_0) + z^{-2}}} \qquad \text{ROC: } |z|>1$$

② Time reversal property:-

If $x(n) \xleftrightarrow{ZT} X(z)$ ROC: $r_1 < |z| < r_2$

Then $x(-n) \xleftrightarrow{ZT} X(z^{-1})$ ROC: $r_2 < |z| < r_1$

Proof:-

$$z[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n) z^{-n}$$

Put $-n = m$

$$= \sum_{m=\infty}^{-\infty} x(m) z^{m}$$

$$= \sum_{m=-\infty}^{\infty} x(m)(z^{-1})^{-m}$$

$$z[x(-n)] = X(z^{-1})$$

$$\boxed{\therefore x(-n) \xleftrightarrow{ZT} X(z^{-1})}$$

$$y(n) = u(n) \xleftrightarrow{ZT} Y(z) = \frac{1}{1-z^{-1}}$$

ROC: $|z|>1$

Apply time reversal property

$$y(-n) \xleftrightarrow{ZT} Y(z^{-1})$$

$$u(-n) \xleftrightarrow{ZT} \frac{1}{1-z}$$

⑤ <u>Time shifting property</u>

If $x(n) \xleftrightarrow{ZT} X(z)$

$$x(n-m) \xleftrightarrow{ZT} z^{-m}X(z)$$

<u>proof:</u>

$$Z[x(n-m)] = \sum_{n=-\infty}^{\infty} x(n-m)\, z^{-n}$$

put $n-m=k$, where m is integer delay

$$= \sum_{k=-\infty}^{\infty} x(k)\, z^{-(k+m)}$$

$$= \sum_{k=-\infty}^{\infty} x(k)\, z^{-m} z^{-k}$$

$$= z^{-m} \sum_{k=-\infty}^{\infty} x(k)\, z^{-k}$$

$$Z[x(n-m)] = z^{-m}X(z)$$

$$\boxed{\therefore x(n-m) \xleftrightarrow{ZT} z^{-m}X(z)}$$

similarly

$$\boxed{x(n+m) \xleftrightarrow{ZT} z^{m}X(z)}$$

⑥ find

$$y(n) = \delta(n) \xleftrightarrow{ZT} Y(z) = 1$$

$$y(n-1) \xleftrightarrow{ZT} z^{-1}Y(z)$$

$$\delta(n-1) \xleftrightarrow{ZT} z^{-1}$$

⑦ find the z-Transform of $x(n) = u(n-1)$

$$y(n) = u(n) \xleftrightarrow{ZT} Y(z) = \frac{1}{1-z^{-1}} \quad \text{ROC: } |z|>$$

$$y(n-1) \xleftrightarrow{ZT} z^{-1}Y(z)$$

$$u(n-1) \xleftrightarrow{ZT} \frac{z^{-1}}{1-z^{-1}} \quad \text{ROC: } |z|>1$$

⑧ find the z-Transform $x(n) = u(n) - u(n-3)$

$$X(z) = \frac{1}{1-z^{-1}} - \frac{z^{-3}}{1-z^{-1}} = \frac{1-z^{-3}}{1-z^{-1}} \quad \text{ROC: } |z|>1$$

④ scaling in z-domain property.

If $x(n) \xleftrightarrow{ZT} X(z)$ ROC: $r_1 < |z| < r_2$

Then $a^n x(n) \xleftrightarrow{ZT} X\left(\frac{z}{a}\right)$ ROC: $|a| r_1 < |z| < |a| r_2$

<u>proof:</u>

$$Z[a^n x(n)] = \sum_{n=-\infty}^{\infty} [a^n x(n)]\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \left(\frac{1}{a^{-n}}\right) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \left(\frac{z}{a}\right)^{-n}$$

$$Z[a^n x(n)] = X\left(\frac{z}{a}\right)$$

$$\boxed{a^n(x(n)) \xleftrightarrow{ZT} X\left(\frac{z}{a}\right)}$$

Put $a=-1$

$$\boxed{(-1)^n x(n) \xleftrightarrow{ZT} X(-z)}$$

④ $x(n) = a^n \cos(\omega_0 n) u(n)$

$$y(n) = \cos(\omega_0 n) u(n) \xleftrightarrow{ZT} Y(z) = \frac{1 - z^{-1}\cos(\omega_0)}{1 - 2z^{-1}\cos(\omega_0) + z^{-2}}$$

ROC : $|z| > 1$

apply time scaling property

$$a^n y(n) \xleftrightarrow{ZT} Y\left(\frac{z}{a}\right)$$

$$a^n \cos(\omega_0 n) u(n) \xleftrightarrow{ZT} \frac{1 - z^{-1}\cos(\omega_0)}{1 - 2az^{-1}\cos(\omega_0) + a^2 z^{-2}} \qquad |z| > |a|$$

⑤ $x(n) = a^n \sin(\omega_0 n) u(n)$

$y(n) = \sin(\omega_0 n) u(n) \xleftrightarrow{ZT} Y(z) =$