Signal:
signal is a function that carries information about any physical phenomina (or) Singal is a function of one or more Independent variables that conveys some information.

⇒ $f(t) \rightarrow$ 1D signal
Ex:- Auduio signal, ECG, EEG, voltage signal, current signal

⇒ $f(x,y) \rightarrow$ 2D signal
Ex:- Image signal

⇒ $f(x_1, x_2 \cdots x_n) \rightarrow$ nd signal
Ex:- space signal

⇒ $f(x,y,t) \rightarrow$ 3D signal
Ex:- Video signal

Types of signal

① continuous vs Discrete time signals

→ continuous-time signal is a signal where 't' is always continuous but amplitude can be either continuous nor discrete. (For con)

→ Discrete time signal is a signal where 't' is decrete but amplitude is continuous such type of signal is said to be Discrete time signals

② Analog vs Digital signals

→ Analog signal is a signal where time and Amplitude is always continuous.

→ Digital signal is a signal where time and Amplitude always discrete.

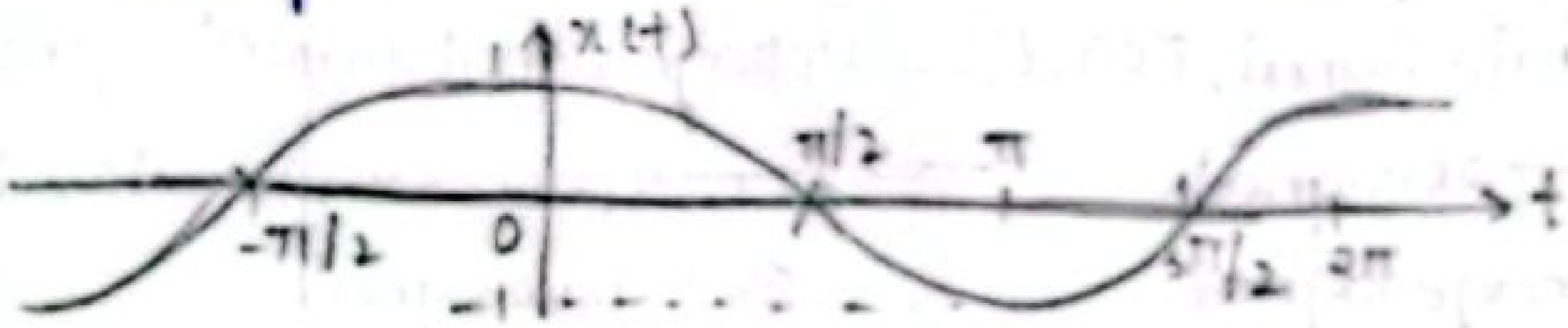
→ All Analog signals are continuous signals but All continuous time signals are not Analog signals

→ All digital signals are discrete signals but All discrete time signals are not digital signals.

③ Even (or) Odd signals $x(t)$

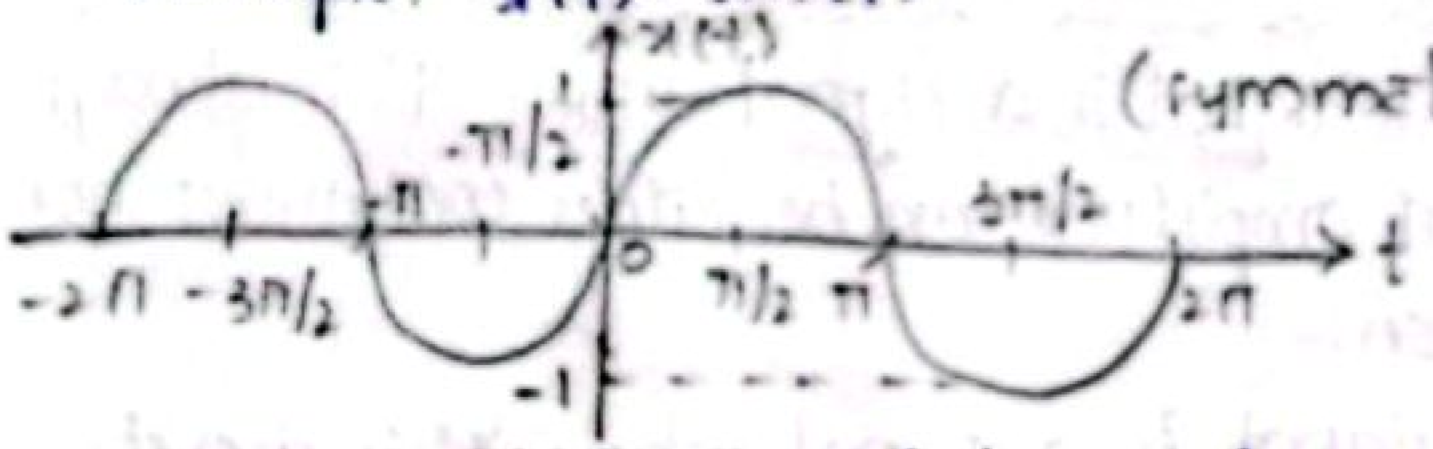
→ The signal $x(t)$ is said to be Even signal only if the signal satisfies the condition $x(-t) = x(t)$. Then it is Even signal.

examples: $x(t) = \cos(t)$ (symmetric about y axis)



→ signal $x(t)$ is said to be a odd signal if and only if $x(-t) = -x(t)$. Then it is said to odd signal

Example: $x(t) = \sin(t)$ (symmetric about origin)



→ for any odd function whose value is always zero at origin.

proof:- $x(-0) = -x(0)$

$x(0) = -x(0)$

$x(0) + x(0) = 0$

$2x(0) = 0$

$x(0) = 0$

→ signal can be decomposed into even component and odd component

$x(t) = x_e(t) + x_o(t)$ —①

$x(-t) = x_e(-t) + x_o(-t)$ (replace t with $-t$)

$x(-t) = x_e(t) - x_o(t)$ —② $\left\{ \begin{array}{l} x_e(-t) = x_e(t) \\ x_o(-t) = -x_o(t) \end{array} \right\}$

① + ②

$x(t) + x(-t) = 2x_e(t)$

$$\boxed{x_e(t) = \frac{x_o(t) + x(-t)}{2}}$$

① − ②

$x_p(t) - x(-t) = 2x_o(t)$

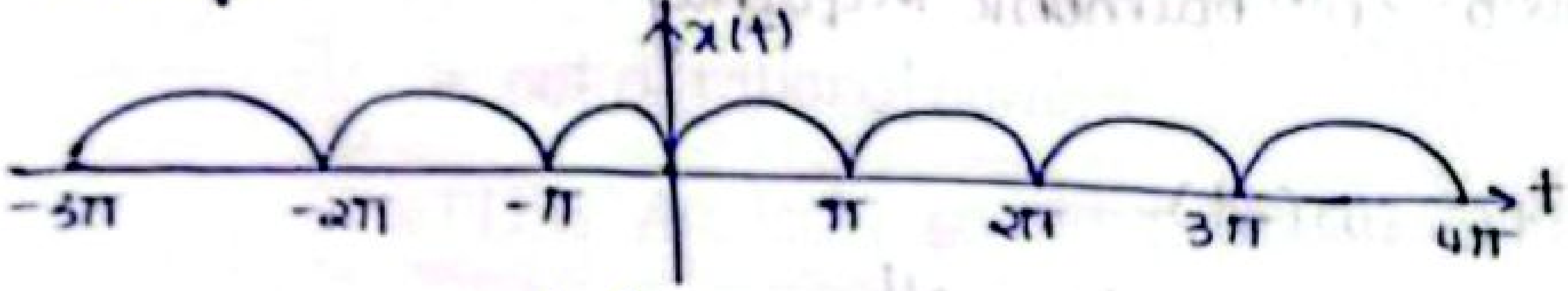
$$\boxed{x_o(t) = \frac{x_e(t) - x(-t)}{2}}$$

Periodic vs Aperiodic signal $x(t)$

→ If the signal $x(t)$ is said to be periodic signal if and only $x(t) = x(t + mT)$ $\forall t$ where m is integer

→ The smallest value of T for which above condition satisfies

example:



period is $\pi, 2\pi, 3\pi, 4\pi$ - - - -

fundamental period is π

Examples

$x(t) = A\sin(\omega_0 t + \phi)$

$x(t + mT) = A\sin(\omega_0(t + mT) + \phi)$

$x(t + mT) = A\sin[\omega_0 t + \omega_0 mT + \phi]$

let $m = 1$

$x(t + T) = A\sin(\omega_0 t + \omega_0 T + \phi)$

If $\omega_0 T = 2\pi$

$x(t + T) = A\sin(\omega_0 t + \phi + 2\pi)$

$\sin(\theta + 2\pi) = \sin\theta$

$x(t+T) = A\sin(\omega_0 t + \phi)$

$x(t+mT) = x(t)$

$\therefore x(t)$ is a periodic signal

$\omega_0 T = 2\pi$

$\boxed{T = \frac{2\pi}{\omega_0}}$ fundamental period

$\omega_0 = \frac{2\pi}{T} \Rightarrow$ fundamental frequency, units (π/sec)

$\omega_0 = 2\pi f_0$

$\boxed{f_0 = \frac{1}{T}}$

ω_0 (fundamental frequency) is 1st harmonic frequency

$2\omega_0 \rightarrow$ 2nd harmonic frequency

$n\omega_0 \rightarrow$ nth harmonic frequency

Ex:

$x(t) = \sin(3t)$

Let compare this with

$x(t) = A\sin(\omega_0 t + \phi)$

$A = 1, \omega_0 = 3, \phi = 0$

$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{3}$

$\therefore$ It is periodic signal

Ex: $x(t) = \cos(\sqrt{2}\pi t)$

$T = \frac{2\pi}{\omega_0} = \frac{2\cancel{\pi}}{\sqrt{2}\cancel{\pi}} = \sqrt{2}$

$x(t) = x(t+mT)$

$\therefore$ It is periodic signal

$\rightarrow$ sum of two periodic signals does not always periodic signal. It is periodic only it satifies some con...

$\rightarrow$ $x_1(t)$ is a periodic signal with the fundamental period T_1 and $x_2(t)$ is a periodic signal with fundamental Period T_2. $x(t) = x_1(t) + x_2(t)$ is periodic. If and if only $\frac{T_1}{T_2} = \frac{m}{n}$ (rational number). The fundamental Period $T_0 = nT_1 = mT_2$ (or) LCM of T_1, T_2; $T_0 = LCM(T_1, T_2)$

Example:-

$x(t) = \sin(3t) + \cos(\sqrt{2}\pi t)$

periodic, periodic

$T_1 = \frac{2\pi}{3}$ $\quad$ $T_2 = \sqrt{2}$

$T = \frac{2\pi}{\omega_0}$

$\frac{T_1}{T_2} = \frac{\frac{2\pi}{3}}{\sqrt{2}} = \frac{\sqrt{2}\pi}{3}$

$\frac{T_1}{T_2}$ is not a rational number

$\therefore x(t)$ is Aperiodic (or) non periodic signal

Example:

$x(t) = \sin(3t) + \cos(5t)$

periodic, periodic

$T_1 = \frac{2\pi}{3}$ $\quad$ $T_2 = \frac{2\pi}{5}$

$\frac{T_1}{T_2} = \frac{\frac{2\pi}{3}}{\frac{2\pi}{5}} = 5/3$

$\frac{T_1}{T_2}$ is rational $\therefore$ it is periodic

Fundamental period is

$\frac{T_1}{T_2} = \frac{5}{3}$

$T_0 = 3T_1 = 5T_2$

$= 3 \cdot \frac{2T_1}{3} = 5 \times \frac{2\pi}{5}$

$T_0 = 2\pi = 2\pi$

$\therefore T_0 = 2\pi$

Example:-

$x(t) = \sin(3t) + \cos(5t) + \sin(7t)$

$\downarrow$ periodic, $\downarrow$ periodic, $\downarrow$ periodic

$T_1 = \frac{2\pi}{3}$, $T_2 = \frac{2\pi}{5}$, $T_3 = \frac{2\pi}{7}$

$\frac{T_1}{T_2} = \frac{5}{3}$, $\frac{T_2}{T_3} = \frac{7}{5}$, $\frac{T_1}{T_3} = 7/3$

$\because x(t)$ is periodic signal

$T_0 := \frac{T_1}{T_2} = \frac{5}{3}$

$T_0 = 3T_1 = 5T_2 = 7T_3$

$\therefore T_0 = 3 \times \frac{2\pi}{3} = 5 \times \frac{2\pi}{5} = 7 \times \frac{2\pi}{7}$

$\therefore T_0 = 2\pi$

Example:-

$x(t) = \sin(\omega_0 t) + \sin(2\omega_0 t)$

$\downarrow$ periodic, $\downarrow$ periodic

$T_1 = \frac{2\pi}{\omega_0}$, $T_2 = \frac{\pi}{\omega_0}$

$\frac{T_1}{T_2} = 2$ (rational number)

$\therefore x(t)$ periodic signal

Fundamental period

$T_0 = T_1 = 2T_2$

$$\boxed{T_0 = \frac{2\pi}{\omega_0}}$$

example:-

$x(t) = \sin(\omega_0 t) + \sin(2\omega_0 t) + \sin(3\omega_0 t)$

$\downarrow$ periodic, $\downarrow$ periodic, $\downarrow$ periodic

$T_1 = \frac{2\pi}{\omega_0}$, $T_2 = \frac{\pi}{\omega_0}$, $T_3 = \frac{2\pi}{3\omega_0}$

$\frac{T_1}{T_2} = 2$, $\frac{T_2}{T_3} = \frac{3}{2}$, $\frac{T_1}{T_3} = 3$

$\therefore$ It is periodic signal

Fundamental period

$T_0 = T_1 = 2T_2 = 3T_3$

$$\boxed{\therefore T_0 = 2\pi}$$

Example:-

$x(t) = \sin(\omega_0 t) + \sin(2\omega_0 t) + \sin(3\omega_0 t) + \sin(4\omega_0 t)$

$\downarrow$ periodic, $\downarrow$ periodic, $\downarrow$ periodic, $\downarrow$ periodic

$T_1 = \frac{2\pi}{\omega_0}$, $T_2 = \frac{\pi}{\omega_0}$, $T_3 = \frac{2\pi}{3\omega_0}$, $T_4 = \frac{\pi}{2\omega_0}$

$\frac{T_1}{T_2} = 2$, $\frac{T_2}{T_3} = \frac{3}{2}$, $\frac{T_1}{T_3} = 3$, $\frac{T_1}{T_4} = 4$, $\frac{T_2}{T_4} = 2$,

$\frac{T_3}{T_4} = \frac{4}{3}$

$\therefore$ It is periodic

Fundamental period is

$T_0 = T_1 = 2T_2 = 3T_3 = 4T_4$

$T_0 = 2\pi$

→ sum of all sinusoidal signals whose frequencies are integers multiples of fundamental frequencies then the resultant signals is periodic signals with the fundamental period of $\frac{2\pi}{\omega_0}$ (same is valid for cosine)

Energy vs power signals

Energy of a given signal $x(t)$ is

$$E=\int_{-\infty}^{\infty}|x(t)|^2dt=\lim_{T\to\infty}\int_{-T}^{T}|x(t)|^2dt$$

Average power of a given signal ($x(t)$) is

$$P_{avg}=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|x(t)|^2dt$$

① Energy is finite and power is ~~infinite~~ zero

then $0<E<\infty$

$P_{avg}=0$

The signal $x(t)$ is Energy signal

$\left.\begin{array}{l}0<P_{avg}<\infty\\ E=\infty\end{array}\right\}$ the signal $x(t)$ is power signal

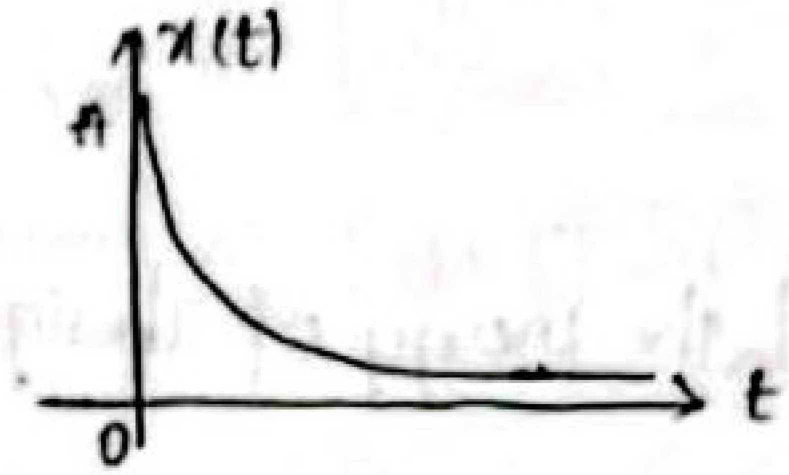
③ If Energy and power are 0 (or) ∞

$E=0$, $P=0$	$E=\infty$, $P=\infty$

neither a Energy nor a power signal

Q check weather the following signals are Energy signals (or) not

① $x(t)=Ae^{-at}u(t)$ Exponentially decaying signal



Energy of a signal $x(t)$

$$E=\int_{-\infty}^{\infty}|x(t)|^2dt\Rightarrow\int_{-\infty}^{0}|x(t)|^2dt+\int_{0}^{\infty}A^2(e^{-at})^2dt$$

$$=\int_{-\infty}^{0}0+\int_{0}^{\infty}A^2(e^{-at})^2dt$$

$$=A^2\int_{0}^{\infty}e^{-2at}dt$$

$$=A^2\left.\frac{e^{-2at}}{-2a}\right|_0^{\infty}$$

$$=\frac{-A^2}{2a}[e^{-\infty}-e^{0}]$$

$$=\frac{-A^2}{2a}(0-1)$$

$$=\frac{A^2}{2a}$$

$$P_{avg}=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|x(t)|^2dt$$

$$=\lim_{T\to\infty}\frac{1}{2T}\int_{0}^{T}A^2(e^{-at})^2dt$$

$$=\lim_{T\to\infty}\frac{A^2}{2T}\left.\frac{e^{-2at}}{-2a}\right|_0^{T}$$

$$=\lim_{T\to\infty}\frac{-A^2}{4Ta}[e^{-2aT}-e^{0}]$$

$$=\lim_{T\to\infty}\frac{A^2}{4Ta}=0$$

$0 < E < \infty$

$P_{avg} = 0$

$\because x(t)$ is a Energy signal, the Energy of the signal

$$E = \frac{A^2}{(2a)}$$

⑤ $x(t) = A\cos(\omega_0 t + \phi)$

Energy signal: $\int_{-\infty}^{\infty} |x(t)|^2 dt$

$$= \int_{-\infty}^{\infty} A^2 \cos^2(\omega t + \phi) dt$$

$$= A^2 \int_{-\infty}^{\infty} \left(\frac{1 + \cos(2\omega t + 2\phi)}{2}\right) dt$$

$$= A^2 \left[\int_{-\infty}^{\infty} \frac{1}{2} dt + \int_{-\infty}^{\infty} \frac{\cos[2(\omega t + \phi)]}{2} dt\right]$$

$$= \frac{A^2}{2} \left[t \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{\cos[2(\omega t + \phi)]}{2} dt\right]$$

$$= \frac{A^2}{2} \left[\infty + \int_{-\infty}^{\infty} \frac{\cos(2(\omega t + \phi))}{2} dt\right]$$

$$= \infty$$

Average power $= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-\infty}^{\infty} A^2 \cos^2(\omega_0 t + \phi) dt$$

$$\lim_{T \to \infty} \frac{A^2}{2T} \int_{-\infty}^{\infty} \frac{1 + \cos(2(\omega_0 t + \phi))}{2} dt$$

$$\Rightarrow \lim_{T \to \infty} \frac{A^2}{4T} \left[dt \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \cos(2\omega_0 t + 2\phi) dT \right]$$

$$\Rightarrow \lim_{T \to \infty} \frac{A^2}{4T} \left[dt \int_{-T}^{T} + \int_{-T}^{T} \cos(2\omega t + 2\phi) dt \right]$$

even function

$$\cancel{\Rightarrow \lim_{T \to \infty} \frac{A^2}{4T} \left[t\, 2T + \left[\frac{\sin(2\omega t + 2\phi)}{2\omega_0}\right]_{-T}^{T} \right]}$$

$$\cancel{\Rightarrow \lim_{T \to \infty} \frac{A^2}{4T} \left[+2T + \frac{\sin(2\omega_0(2T) + 2\phi)}{2\omega_0} \right]}$$

$$\cancel{\Rightarrow \lim_{T \to \infty} \frac{A^2}{4T} [2T +)}$$

$$\Rightarrow \lim_{T \to \infty} \frac{A^2}{4T} \left[2T + 2 \int_0^T \cos(2\omega_0 T + 2\phi) dt \right]$$

$$\Rightarrow \lim_{T \to \infty} \left[\frac{A^2}{4T} 2T + \frac{A^2}{4T}(2) \left[\frac{\sin 2\omega_0 T + 2\phi}{2\omega_0} \right]_0^T \right]$$

$$\Rightarrow \lim_{T \to \infty} \left[\frac{A^2}{2} + \frac{A^2}{2T} \frac{1}{2\omega_0} [\sin 2\omega_0 T_{+2\phi} - \sin 2\omega_0(0) + 2\phi] \right]$$

$$\Rightarrow \lim_{T \to \infty} \left[\frac{A^2}{2} + \frac{A^2}{4T\omega_0} [\sin 2\phi] \right]$$

$$\Rightarrow \lim_{T \to \infty} \frac{A^2}{2} + \lim_{T \to \infty} \frac{A^2}{4T\omega_0} \sin 2\phi$$

$$\Rightarrow \frac{A^2}{2}$$

Energy is ∞, power is finite.

$\therefore x(t)$ is a power signal

NOTE:

RMS value $= \sqrt{P_{avg}}$

$\because$ for $x(t) = A\cos(\omega_0 t + \phi)$ RMS $= \frac{A}{\sqrt{2}}$

similarly $x(t) = A\sin(\omega_0 t + \phi)$, then Energy is ∞

Average power is $A^2/2$ $\therefore$ $x(t)$ is power signal

Deterministic vs random signals

→ Deterministic signal is a signal whose values can be determined at any instant of time 't' such signals are Deterministic signals. Can be expressed mathematically

Example:- $\sin(t)$, e^{t} etc

→ Random signal is a signal whose value can't be determined at any instant of time 't'. Random signals are held with / Analysed with probability manner.

Example:- noise signal

causal, Anti causal, non-causal signals

→ causal is a signal if $x(t)=0$, for $t<0$

→ Anti causal signal, if $x(t)=0$ for $t>0$

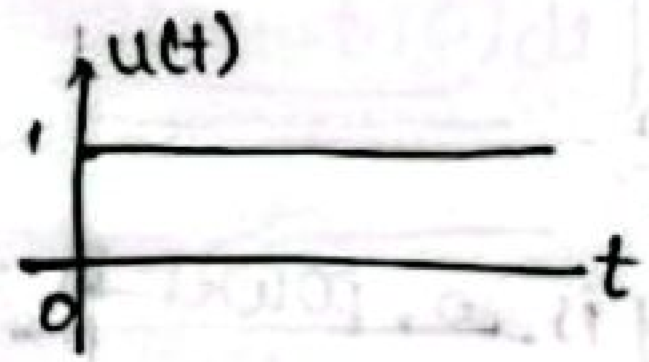
→ non-causal signal, if $x(t)=0$ for ~~t>0~~ t (or) Both $t>0, t<0$

Standard continuous time signals.

Basic signals

① **unit step signal**

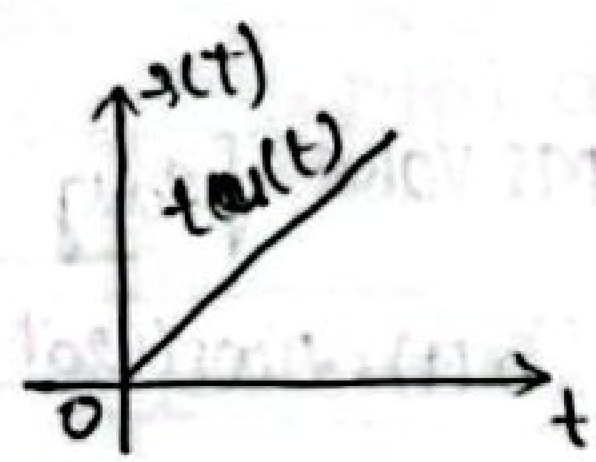
$$u(t) = \begin{cases} 1 & t \text{ (or) } t>0 \\ 1/2 & t \text{ (or) } t=0 \\ 0 & t \text{ (or) } t<0 \end{cases}$$



It is a causal signal

② **Ramp signal**

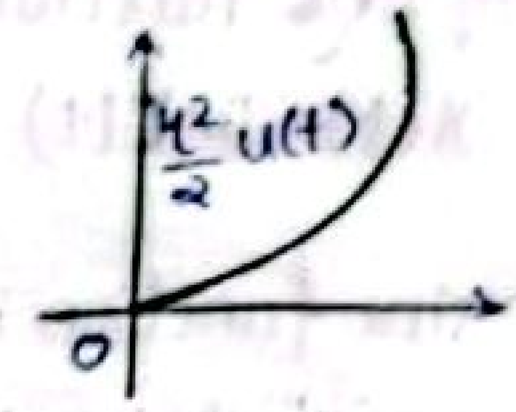
$$x(t) = \begin{cases} t & t \text{ (or) } t\geq 0 \\ 0 & t \text{ (or) } t<0 \end{cases}$$



$x(t) = t\, u(t)$

③ **parabolic signal**

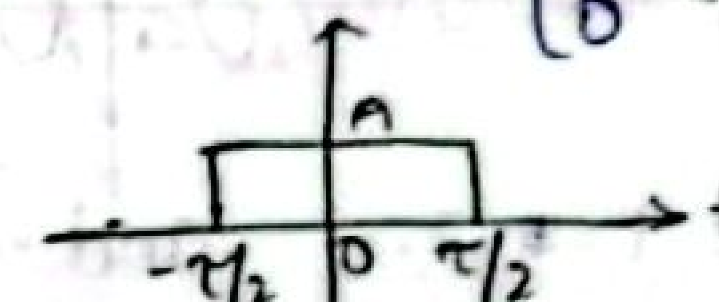
$$P(t) = \begin{cases} t^2/2 & t \text{ (or) } t>0 \\ 0 & t \text{ (or) } t<0 \end{cases}$$



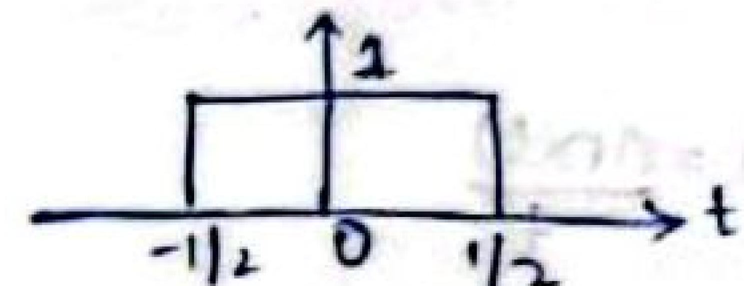
$P(t) = \frac{t^2}{2} u(t)$

④ **Rectangular signal (or) pulse**

$$x(t) = A\,\text{rect}(t/\tau) = A\Pi(t/\tau) = \begin{cases} A & t \text{ (or) } -\tau/2 \leq t \leq \tau/2 \\ 0 & \text{o} \end{cases}$$



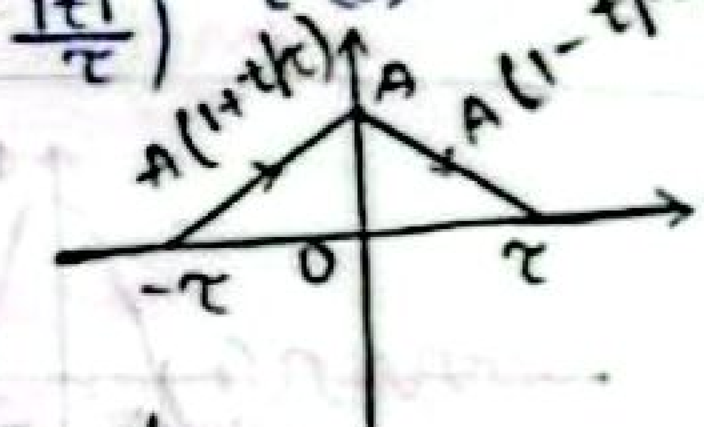
Example:- $x(t) = \text{rect}(t)$ sketch graph



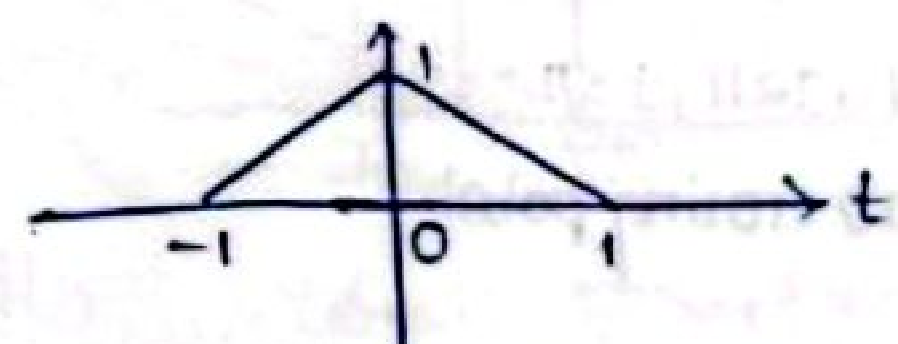
⑤ **Triangular signal (or) Triangular pulse**

$$A\Delta(t/\tau) = \begin{cases} A\left(1-\frac{|t|}{\tau}\right) & t \text{ (or) } -\tau \leq t \leq \tau \\ 0 & \end{cases}$$

Triangular symbol

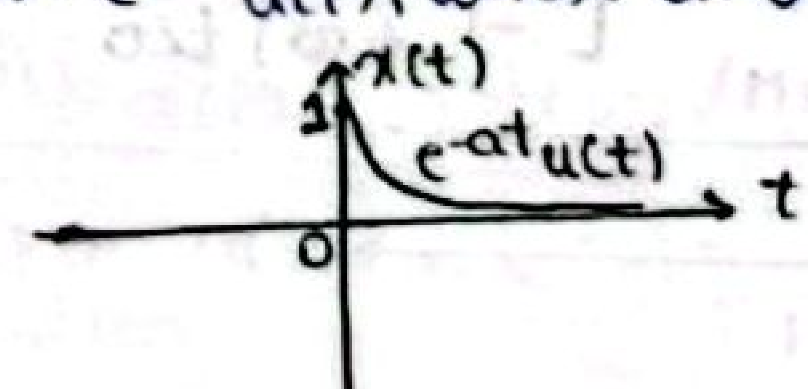


Example:- $x(t) = 1-|t|$ sketch graph



⑥ **Exponentially decaying signal**

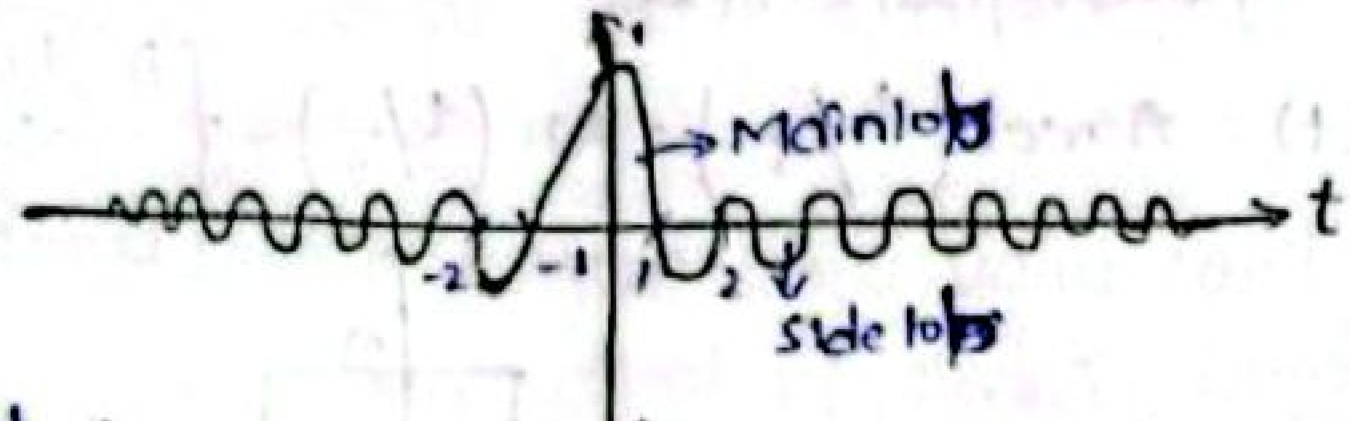
$x(t) = e^{-at}u(t)$, where $a>0$



⑦ sinc function

$$x(t) = \text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

sinc function is product of two signals that is $\sin(\pi t)$ and $1/(\pi t)$

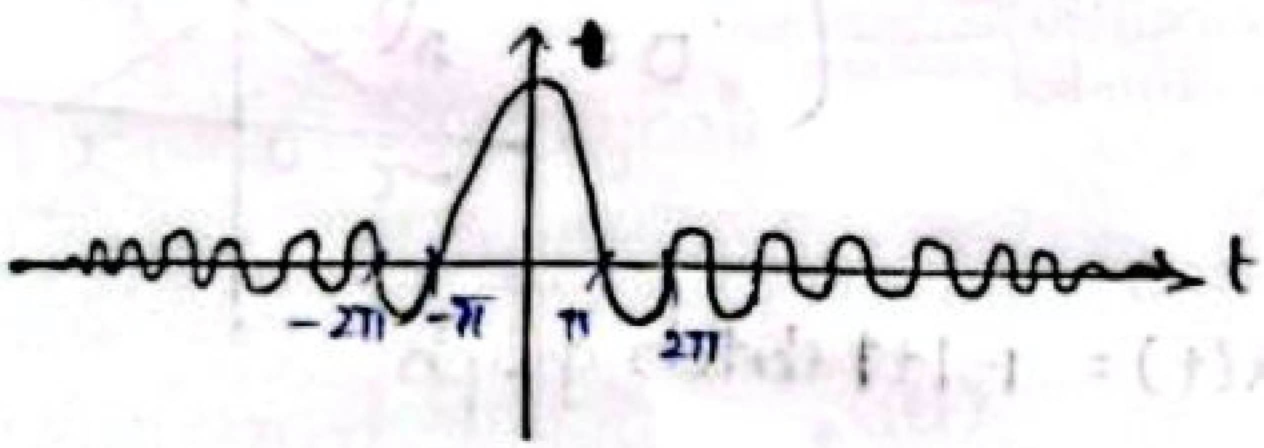


at $t = 1, 2, \cdots$ and $-1, -2 \cdots$ are zero crossing points because at this point it value is zero

⑧ sampling function

$$x(t) = sa(t) = \frac{\sin(t)}{t}$$

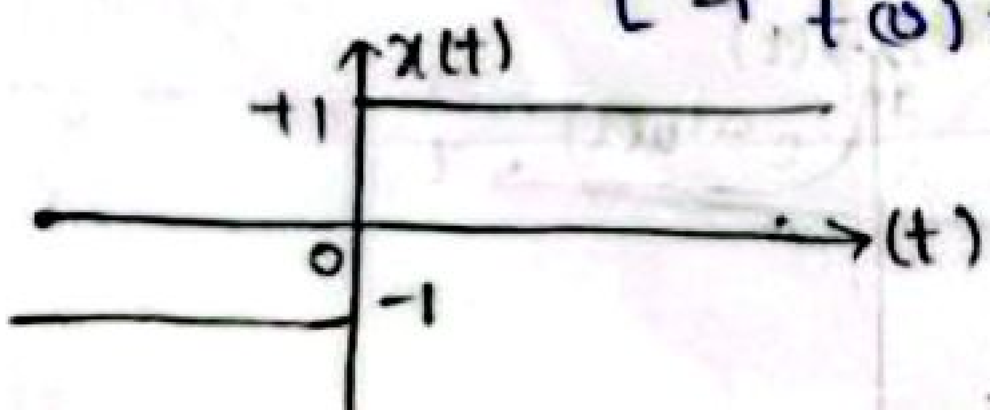
$$\boxed{sa(\pi t) = \text{sinc}(t) = \frac{\sin \pi t}{\pi t}}$$



$t = \pm\pi, \pm 2\pi, \pm 3\pi \cdots$ are zero crossing points

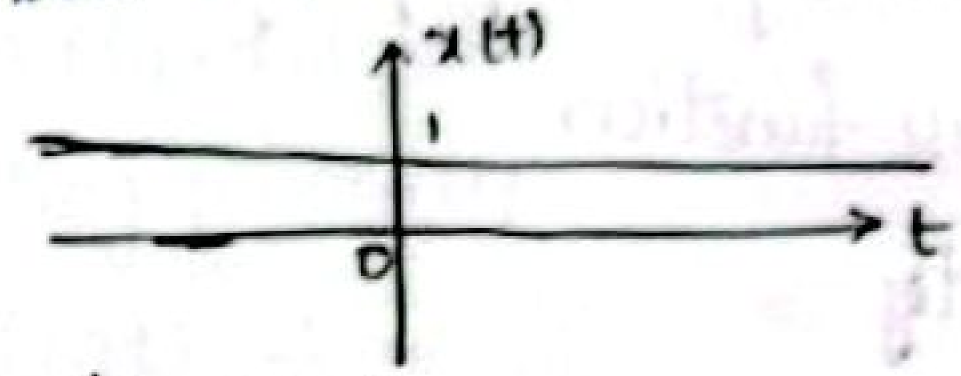
⑨ signum function

$$x(t) = \text{sgn}(t) = \frac{t}{|t|} = \begin{cases} +1 & t \text{ (or) } t > 0 \\ 0 & t \text{ (or) } t = 0 \\ -1 & t \text{ (or) } t < 0 \end{cases}$$



⑨ DC signal

$$x(t) = 1 \;\forall\; t$$



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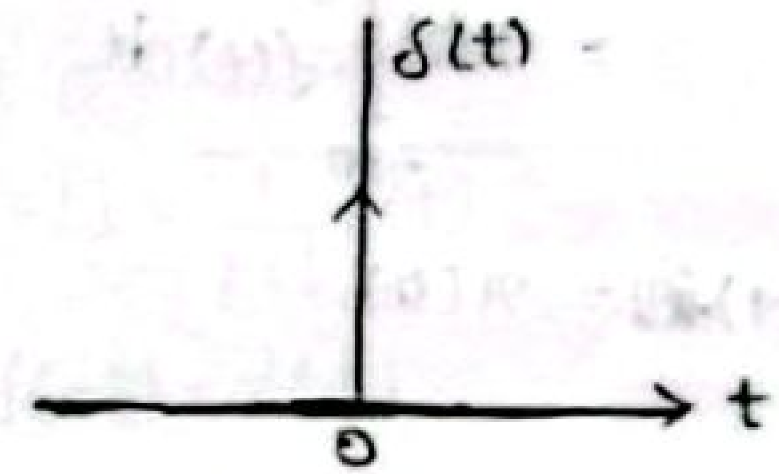
⑩ Impulse signal

Impulse signal is also know as dirac delta function

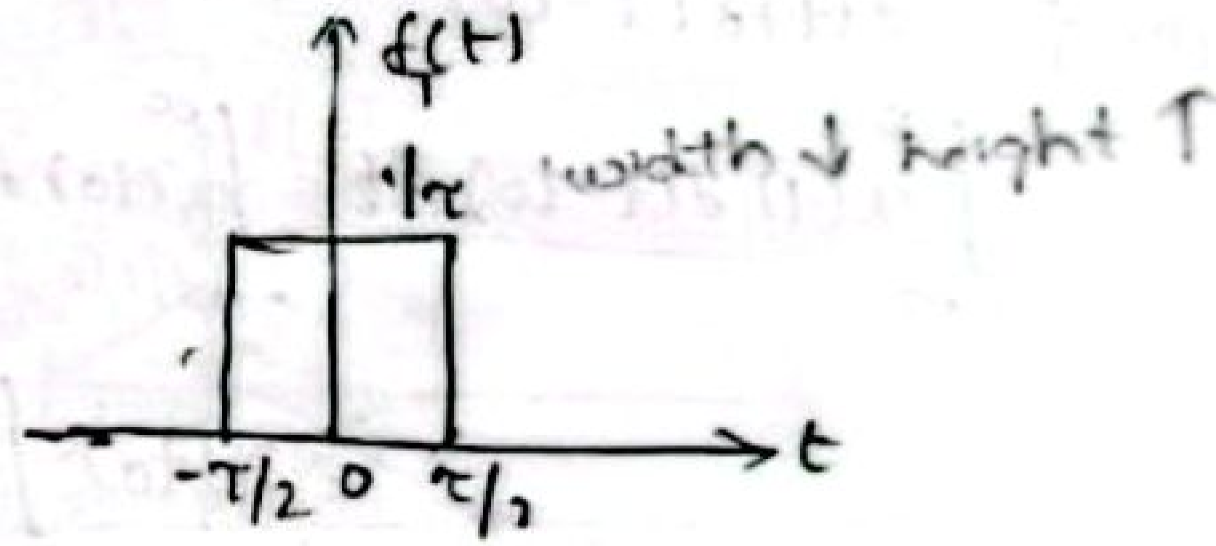
i) $\delta(t) = \begin{cases} \infty & t \text{ (or) } t = 0 \\ 0 & \text{otherwise} \end{cases}$

$$= \int_{-\infty}^{\infty} \delta(t) \text{ (or) } dt = 1$$

when we dealing with Impulse signal we deal with area not with amplitude



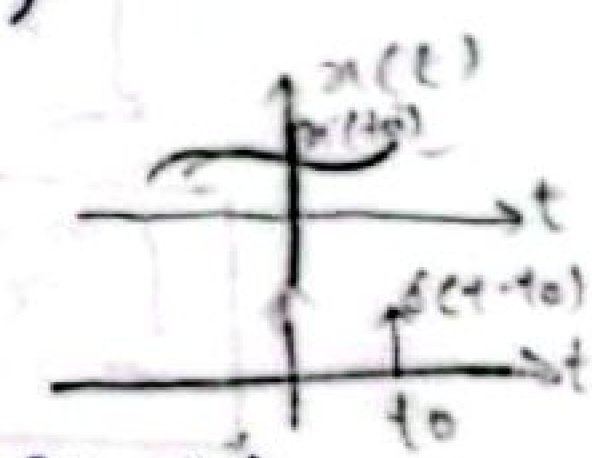
Impulse signal can be understand by rectangular pulse



(ii) $\delta_\tau(t) = \frac{1}{\tau} \text{rect}(t/\tau)$

$$\delta(t) = \lim_{\tau \to 0} \delta_\tau(t)$$

(iii) $x(t)\delta(t) = x(0)\delta(t)$



right shifting operation $x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$

Any signal multiplied with impluse signal than is impluse signal scaled by some property

Properties of Impluse function

① sampling property

ⓐ $\int_{-\infty}^{\infty} x(t)\,\delta(t) = x(0)$

proof:- $x(t)\,\delta(t) = x(0)\,\delta(t)$

$$\int_{-\infty}^{\infty} x(t)\,\delta(t)\,dt = \int_{-\infty}^{\infty} x(0)\,\delta(t)\,dt$$

$$= x(0)\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

$$\int_{-\infty}^{\infty} x(t)\,\delta(t) = x(0)$$

$$\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt = x(t_0)$$

proof:- $x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$

$$\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt = \int_{-\infty}^{\infty} x(t_0)\,\delta(t-t_0)\,dt$$

$$= x(t_0)\int_{-\infty}^{\infty} \delta(t-t_0)\,dt$$

$$\therefore \int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt = x(t_0)$$

$$\boxed{\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt = x(t_0) = x(t)\big|_{t=t_0}}$$

Example: $\int_{-\infty}^{\infty} x(t-t_1)\,\delta(t-t_2)\,dt$

$= x(t-t_1)\big|_{t=t_2}$

$= x(t_2-t_1)$

② Scaling property

ⓐ $\delta(at) = \frac{1}{|a|}\delta(t)$

ⓑ $\delta(at+b) = \delta\left[a\left(t+\frac{b}{a}\right)\right]$

$$= \frac{1}{|a|}\,\delta\left(t+\frac{b}{a}\right)$$

③ Even function

$\delta(at) = \frac{1}{|a|}\delta(t)$

let $a = -1$

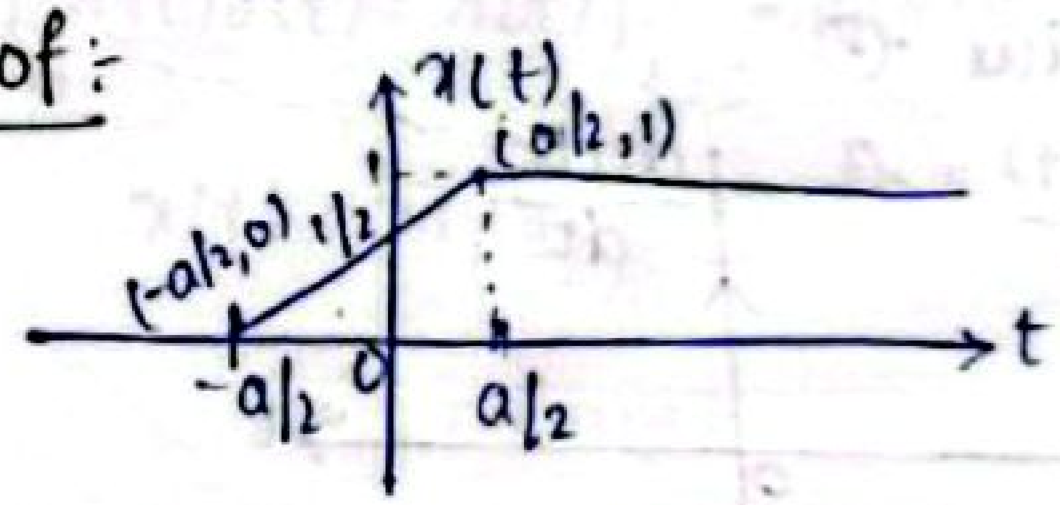
$\delta(-t) = \frac{1}{|-1|}\delta(t)$

$\delta(-t) = \delta(t)$

④

$$\delta(t) = \frac{du(t)}{dt}$$

proof:-



line equation $y = mx + c$

$$m = \frac{y_2-y_1}{x_2-x_1} = \frac{1}{a}$$

$y = \frac{1}{a}x + c$

$x(t) = \frac{1}{a}t + c$

let $t=0$

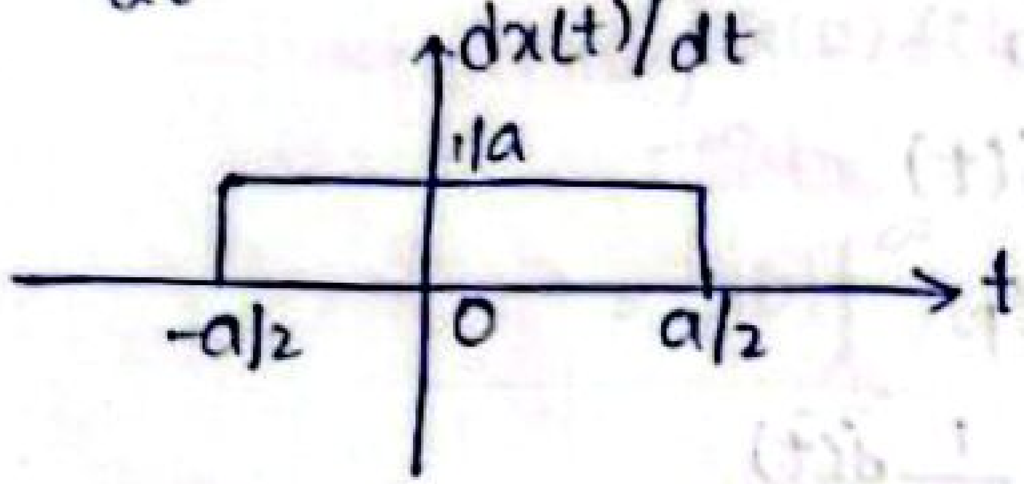
$x(0) = 0 + c$

$x(0) = \frac{1}{2} = c$

$\because x(t) = \begin{cases} \frac{1}{a}t + \frac{1}{2} & (\text{for } -a/2 \le t \le a/2) \\ 1 & (\text{for } t \ge a/2) \end{cases}$

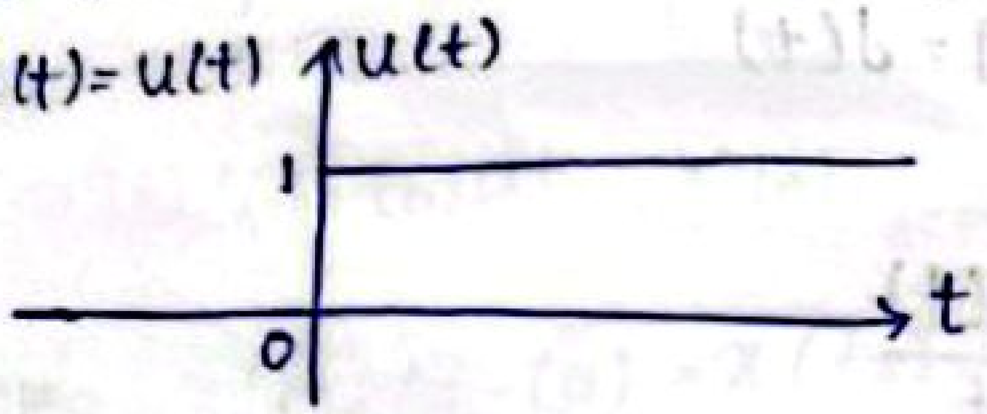
differentating the signal

$\frac{dx(t)}{dt} = \frac{1}{a}$ for $-\frac{a}{2} \le t \le a/2$



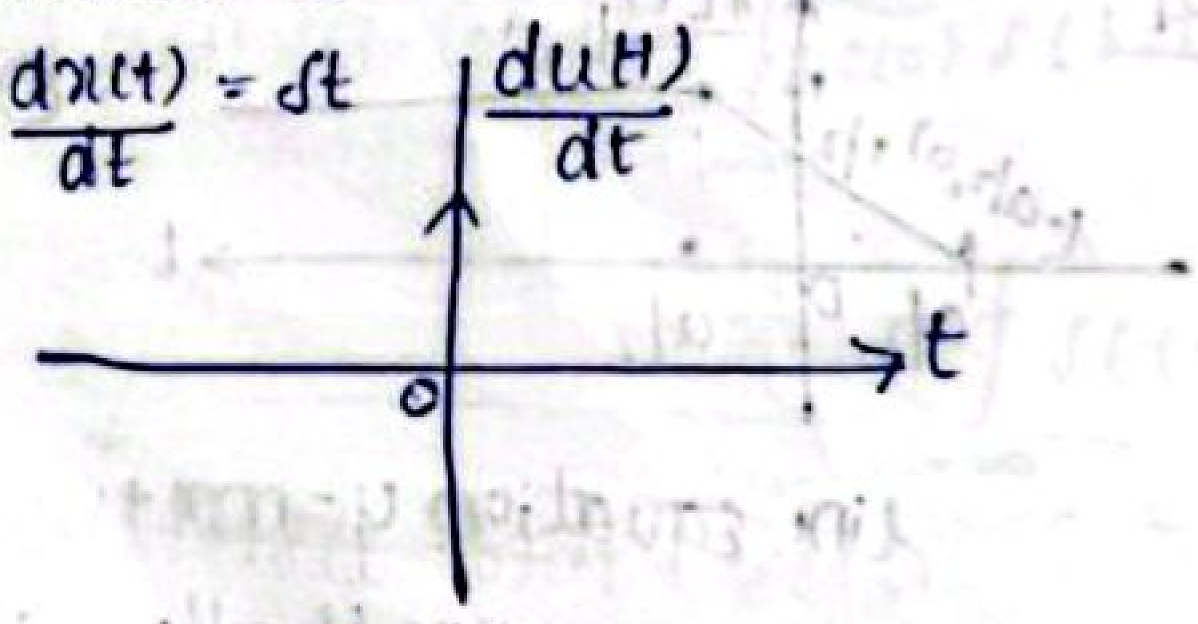
$a \to 0$ This become unit step signal -①

$\lim_{a \to 0} x(t) = u(t)$



It's derivative -②

$\lim_{a \to 0} \frac{dx(t)}{dt} = \delta t$



$\delta(t) = \frac{du(t)}{dt}$

If we integrate impluse we get unit step signal

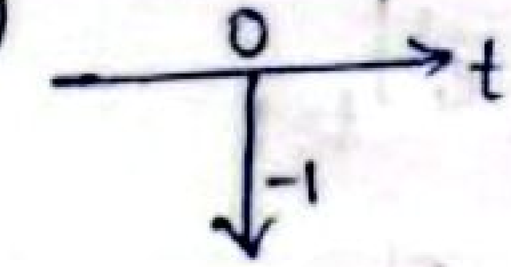
$u(t) = \int^{t} \delta(t)\,dt$

③ $\left[\sin\left(t^2 - \frac{\pi}{2}\right)\right]\delta(t)$

$x(t)\delta(t) = x(0)\delta(t)$ (property)

let replace t with 0

$\sin\left(0 - \frac{\pi}{2}\right)\delta(t) = \sin\left(-\frac{\pi}{2}\right)\delta(t)$



$= (-1)\,\delta(t)$

$= -\delta(t)$

④ $\int_{-\infty}^{\infty} \delta(t-2)\cos\left(\frac{\pi t}{4}\right) dt$

$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)\,dt = x(t)\big|_{t=t_0}$ (property)

$\int_{-\infty}^{\infty} \delta(t-2)\underbrace{\cos\left(\frac{\pi t}{4}\right)}_{x(t)} dt = x(t)\big|_{t=2}$

$\therefore x(t)\big|_{t=2} = \cos\frac{\pi t}{4}\Big|_{t=2} = \cos\frac{\pi(2)}{4_2}$

$= \cos\frac{\pi}{2} = 0$

⑤ $\int_{-\infty}^{\infty} \delta(t)\cos\left(\frac{3t}{2}\right) dt$

$\int_{-\infty}^{\infty} x(t)\delta(t) = x(t)\big|_{t=0} = x(0)$ (property)

$x(t) = \cos\left(\frac{3t}{2}\right)$

$x(0) = \cos(0) = 1$

⑥ $\int_{-\infty}^{\infty} e^{-t}\delta(2t-2)\,dt$

$\delta(at+b) = \frac{1}{|a|}\delta\left(t + \frac{b}{a}\right)$

$\delta(2t-2) = \frac{1}{|2|}\delta\left(t - \frac{2}{2}\right)$

$= \frac{1}{2}\delta(t-1)$

$$\int_{-\infty}^{\infty} e^{-t} \frac{1}{2} \delta(t-1)\,dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{-t}\,\delta(t-1)\,dt \qquad \int_{-\infty}^{\infty} x(t)\,\delta(t-t_0) = x(t)\big|_{t=t_0}$$

$$= \frac{1}{2} e^{-t}\Big|_{t=1}$$

$$= \frac{1}{2} e^{-1}$$

$$= \frac{1}{2e}$$

⑧ $$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^2 e^{-t^2/2}\, \delta(1-2t)\,dt$$

By scaling property

$$\delta(1-2t) = \frac{1}{|2|}\,\delta\left(t-\frac{1}{2}\right)$$

$$= \frac{1}{2}\,\delta\left(t-\frac{1}{2}\right)$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^2 e^{-t^2/2}\, \frac{1}{2}\,\delta\left(t-\frac{1}{2}\right)dt$$

$$\frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} t^2 e^{-t^2/2}\, \delta\left(t-\frac{1}{2}\right)dt$$

$$\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0) = x(t)\Big|_{t=t_0} \text{ (property)}$$

$$\frac{1}{2\sqrt{2\pi}}\, t^2 e^{-t^2/2}\Big|_{t=1/2}$$

$$\frac{1}{2\sqrt{2\pi}}\,\frac{1}{4}\,e^{-\frac{1/4}{2}}$$

$$= \frac{1}{8\sqrt{2\pi}}\, e^{-1/8}$$

$$= \frac{1}{8\sqrt{2\pi}\, e^{1/8}}$$

Basic operations on signals

① Addition of signals

$x_1(t)$

$x_2(t)$

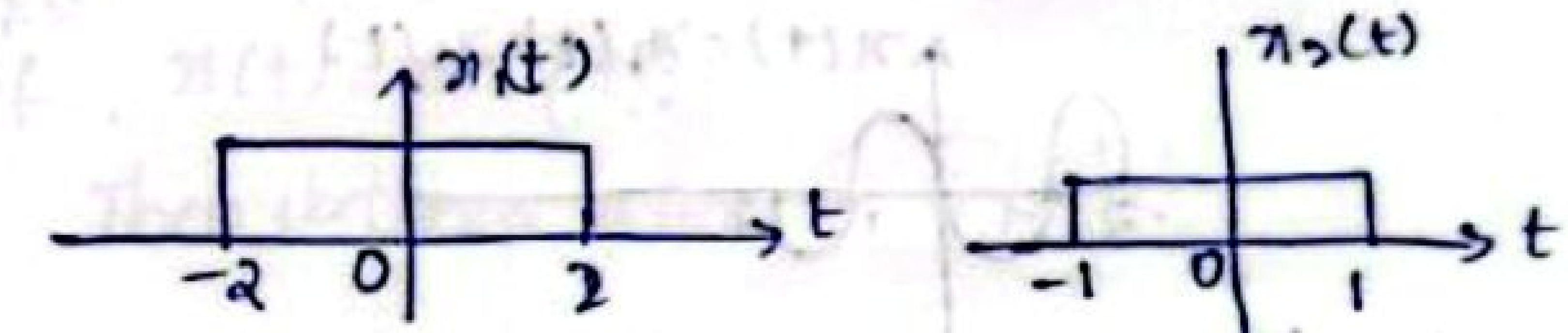
$x(t) = x_1(t) + x_2(t)$

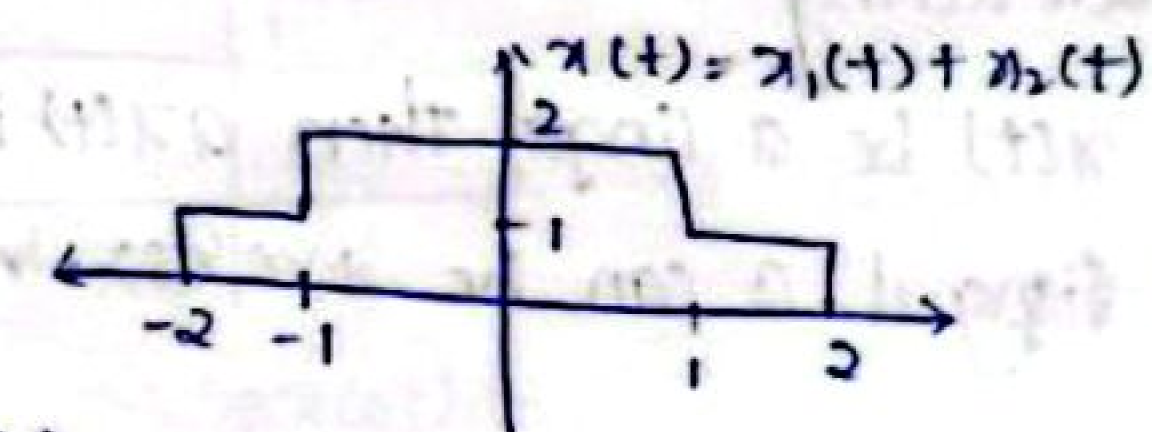
$x(0) = x_1(0) + x_2(0)$

$x(0.1) = x_1(0.1) + x_2(0.1)$

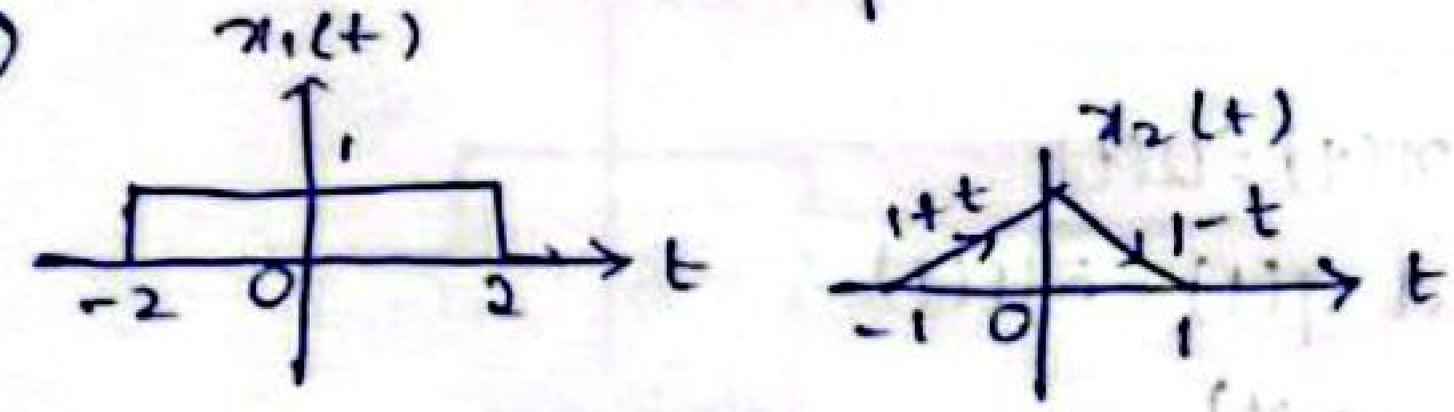
Ex:-

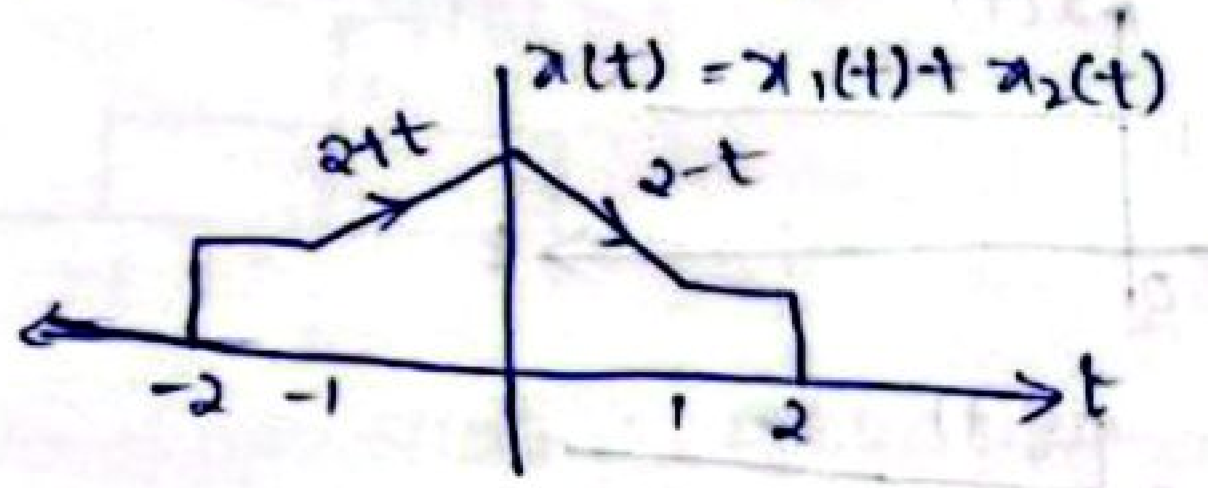
①





②





③ multiplication of signals.

Let $x_1(t)$ and $x_2(t)$ are two signals

$x(t) = x_1(t)\, x_2(t)$

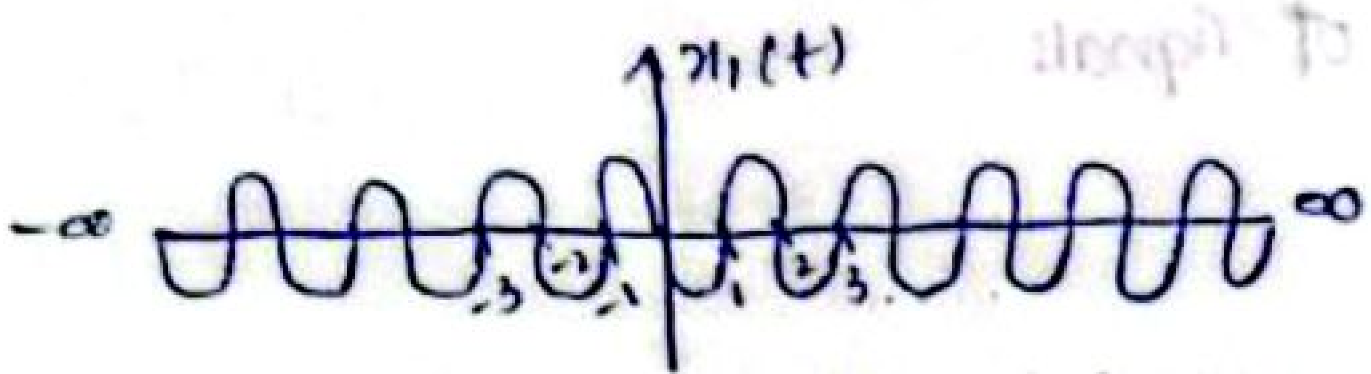
$x(0) = x_1(0) \cdot x_2(0)$

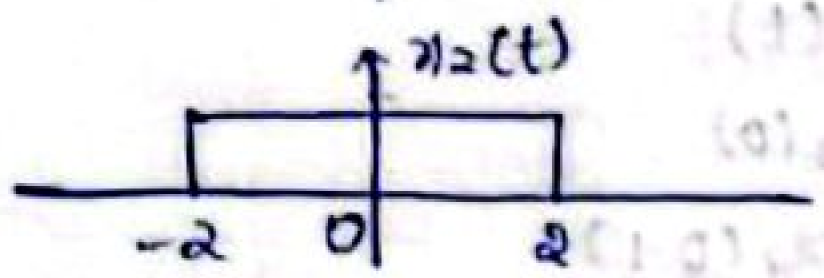
Example:-

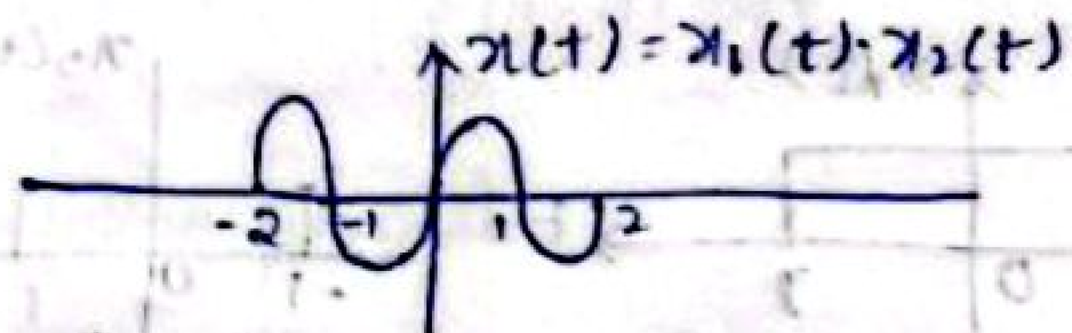
$x_1(t) = \sin(\pi t)$

$x_2(t) = rect(t/4)$

$x(t) = x_1(t) \cdot x_2(t)$







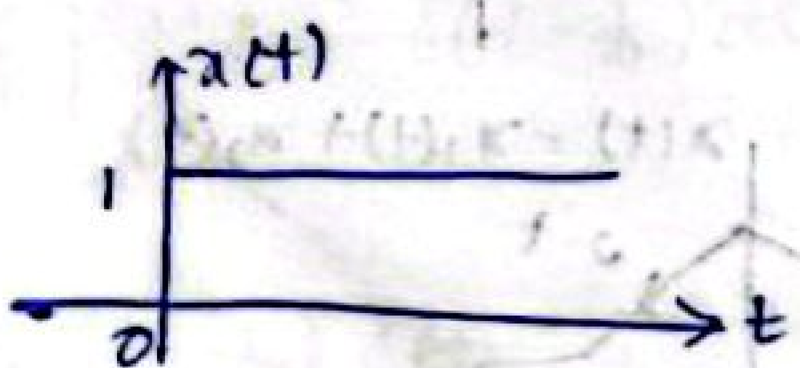
③ Amplitude scaling.

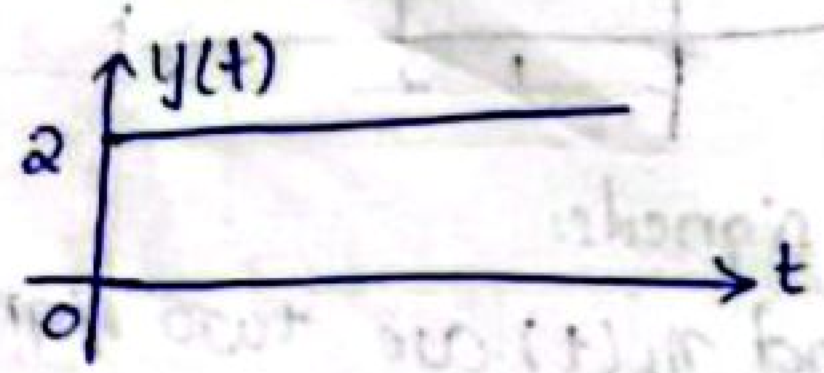
Let $x(t)$ be a singal then $ax(t)$ is Amplitude scaling siggal a can be +ve (or) -ve

Example:

① $x(t) = u(t)$

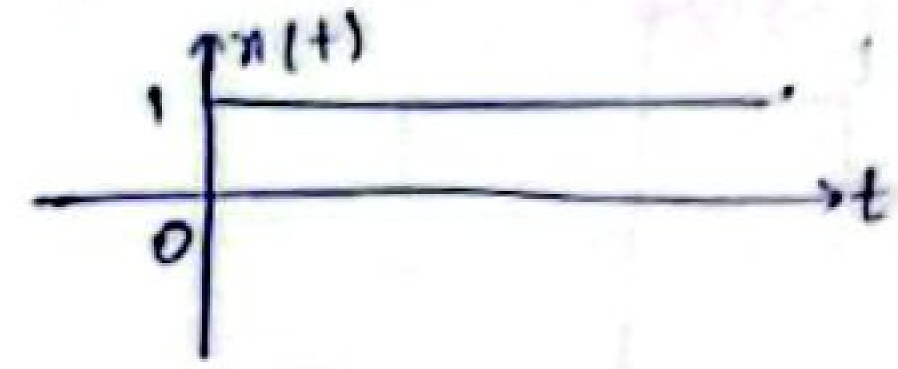
Let $y(t) = 2u(t)$.

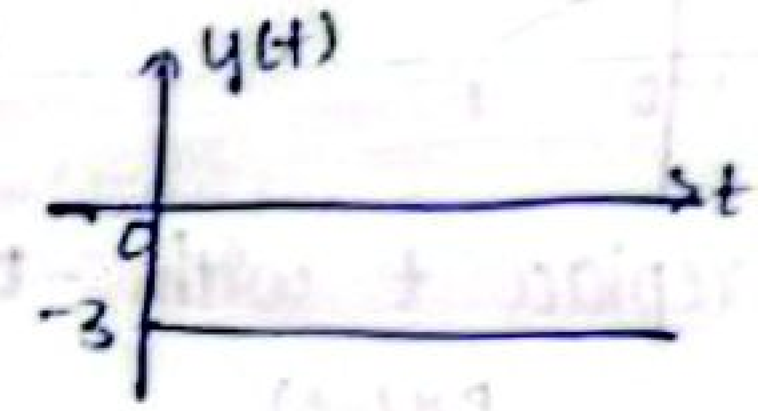




② $x(t) = u(t)$

$y(t) = -3u(t)$





④ Time scaling

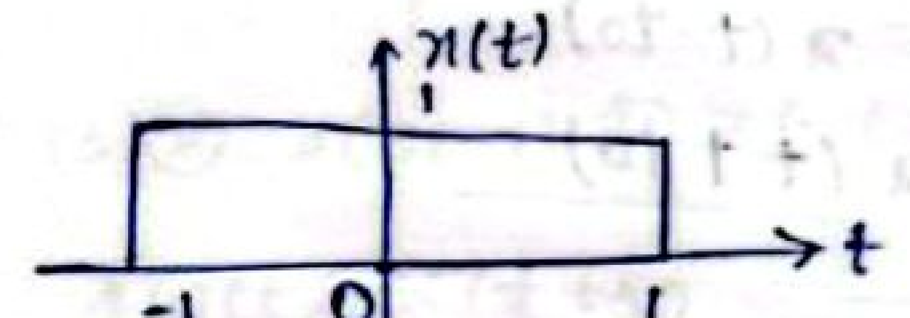
Let $x(t)$ be orginal singal then $x(at)$ be Time scaling siggal. if $a>1$, signal is compressed and $a<1$ then signal is Expand

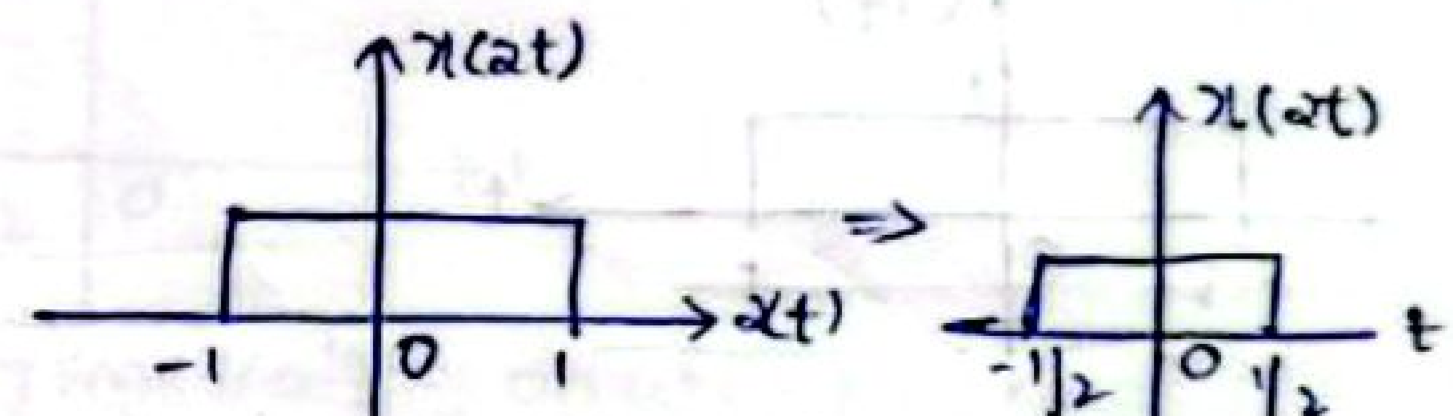
Example:

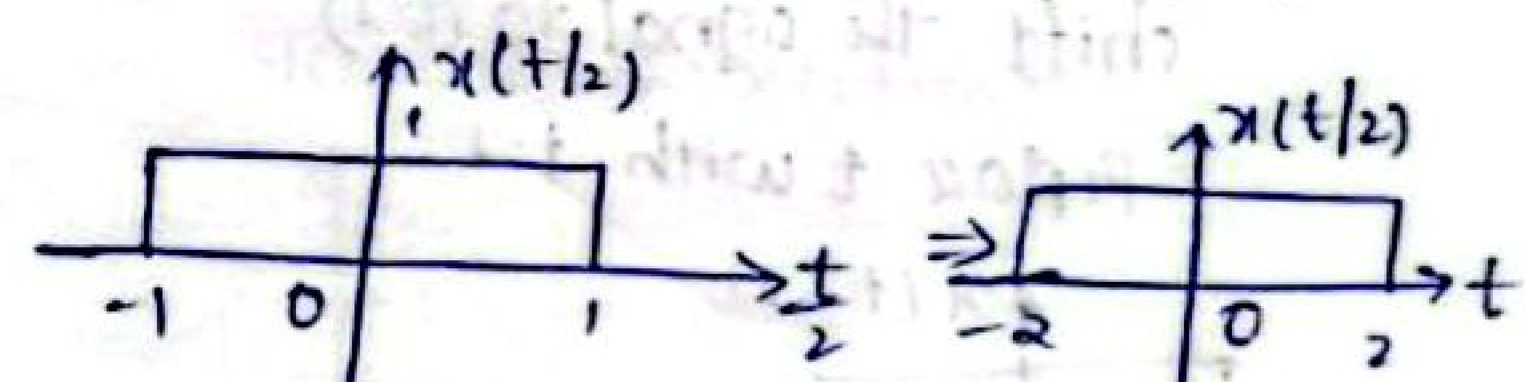
if, $x(t) = rect(t/2)$

Then sketch ⓐ $x(2t)$ ⓑ $x(t/2)$

ⓐ



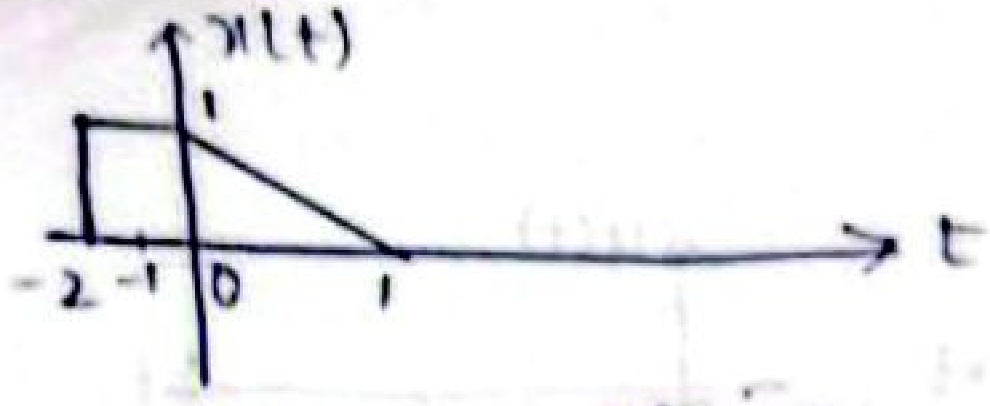




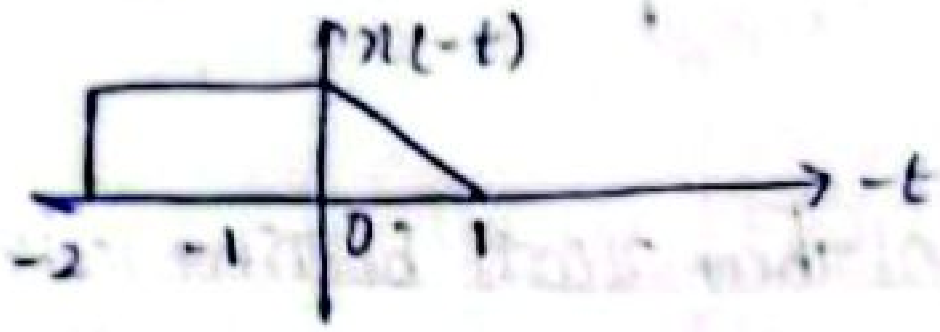
⑤ Time reversal operation (or) Fobling operation

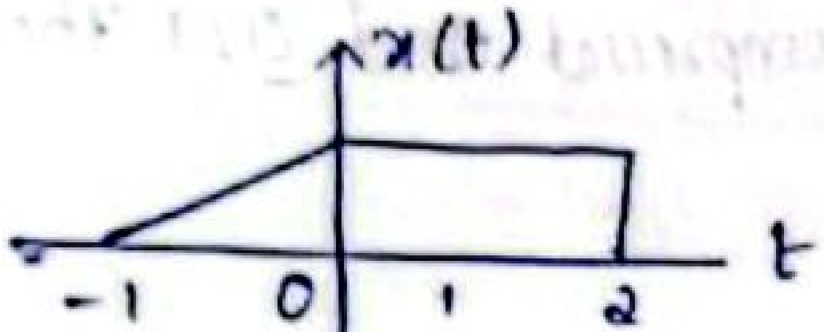
If $x(t)$ is a orginal signal the time reversal version of the signal is $x(-t)$

Example:-



replace t with -t





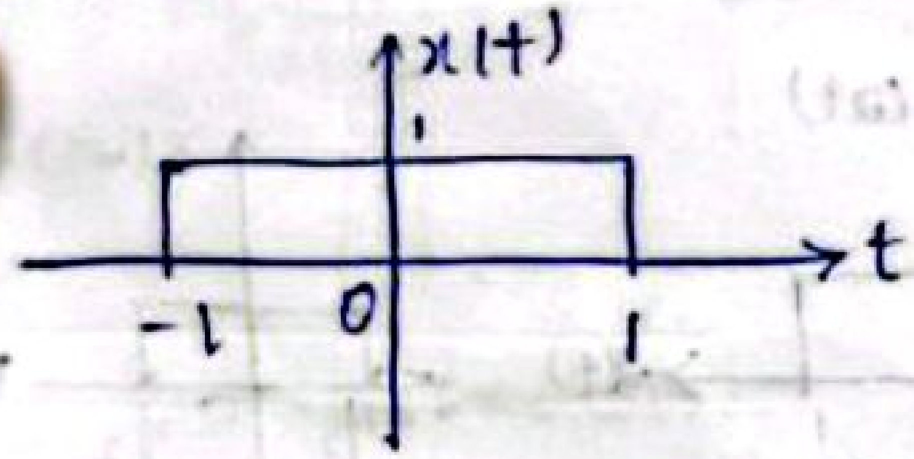
⑥ Time shifting operations:

Two types

① Right shifting $x(t-t_0)$

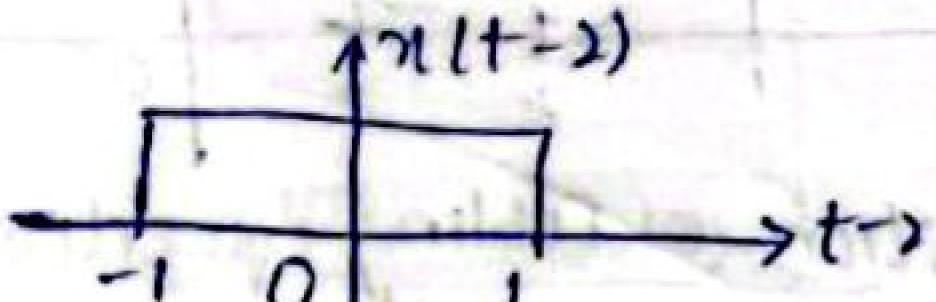
② left shifting $x(t+t_0)$

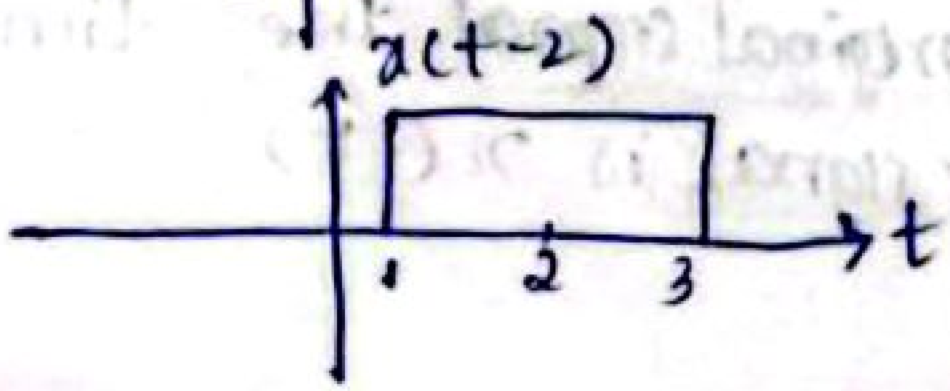
Example:



shift the signal to (t-2)

∵ Replace t with t-2

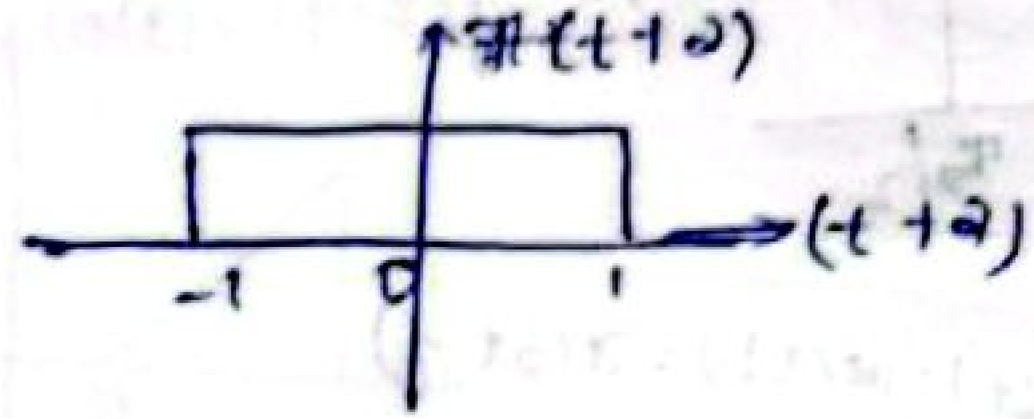


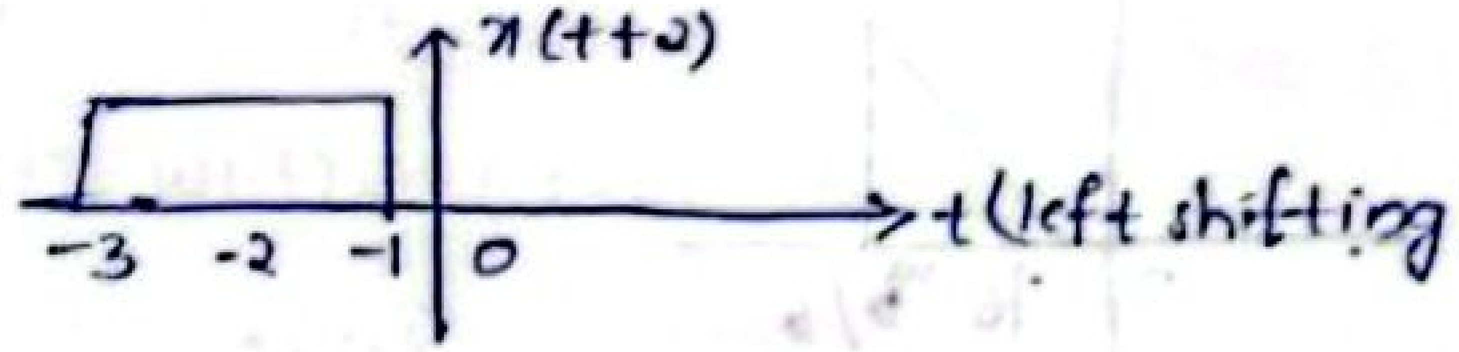


Right shifting signal

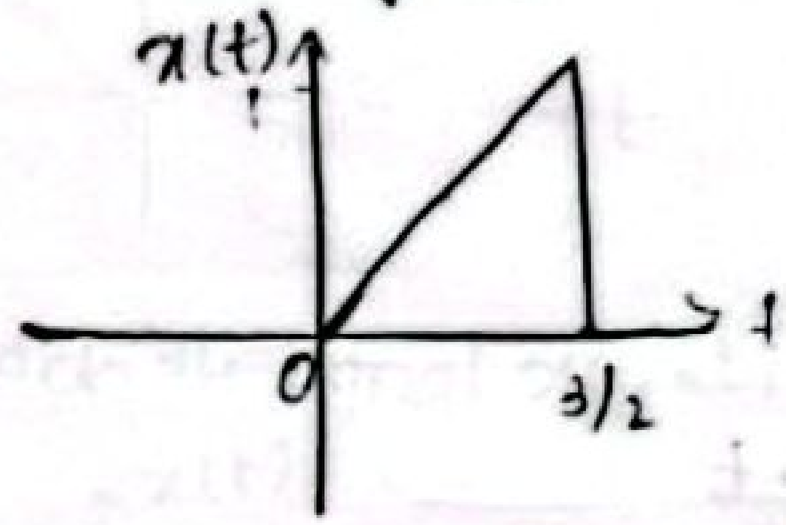
shift the signal to x(t+2)

∵ Replace t with t+2



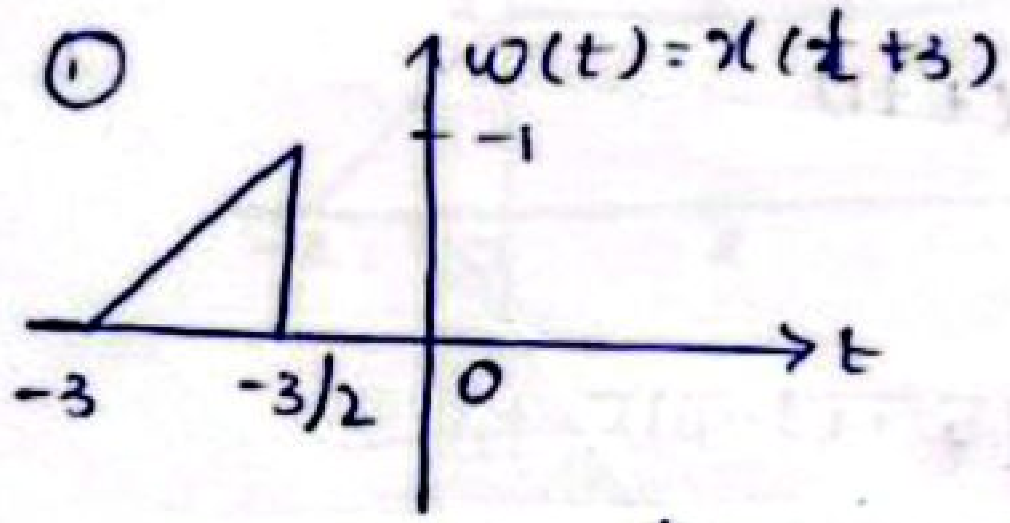


⑦ Sketch the following signals



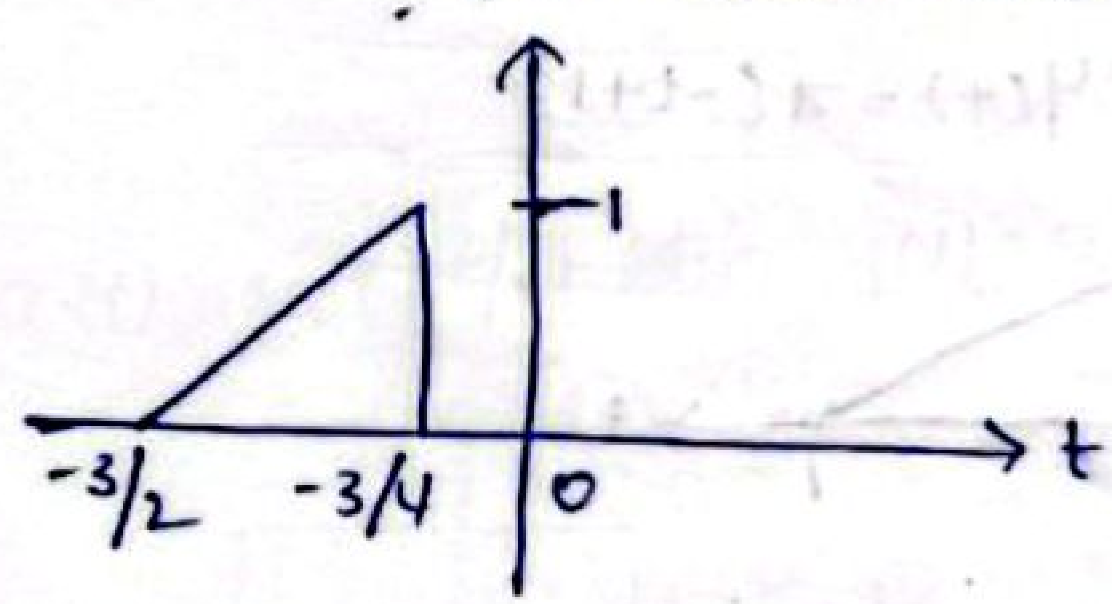
① $x(2t+3)$ ② $x(3t-1)$ ③ $x(-t+1)$ ④ $x(-t-2)$

①

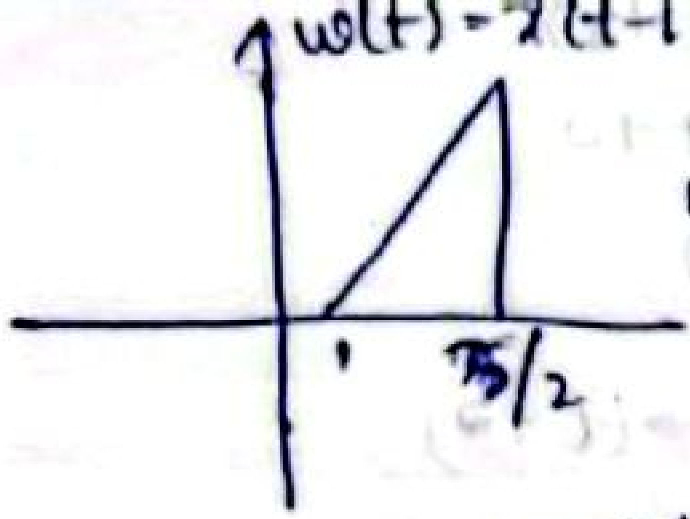


Time scaling on w(t)

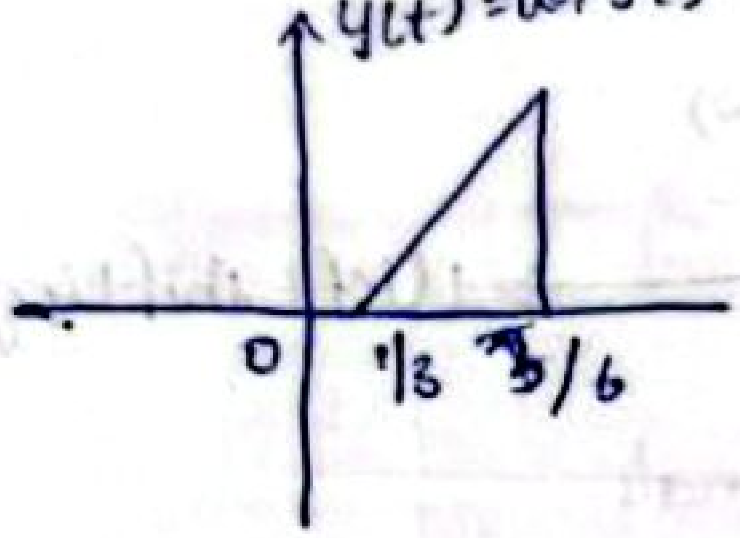
$y(t) = w(2t) = x(2t+3)$



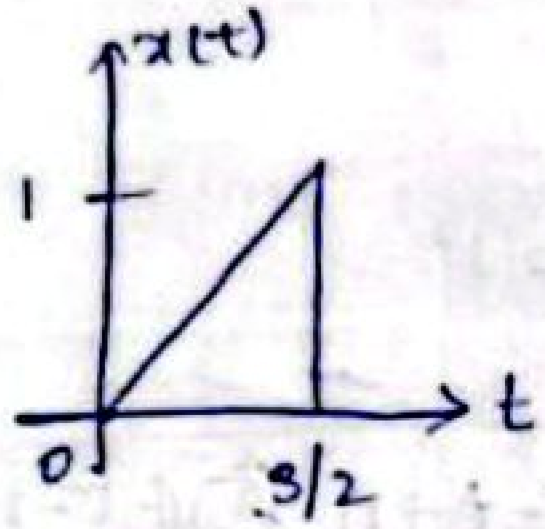
②



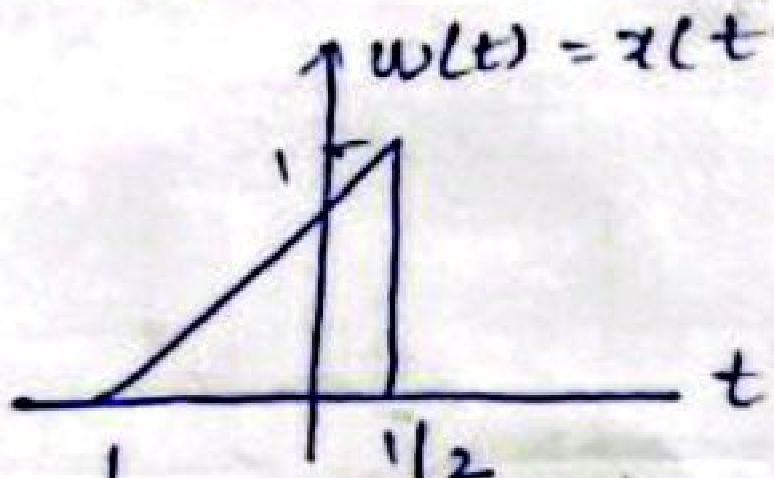
shifting



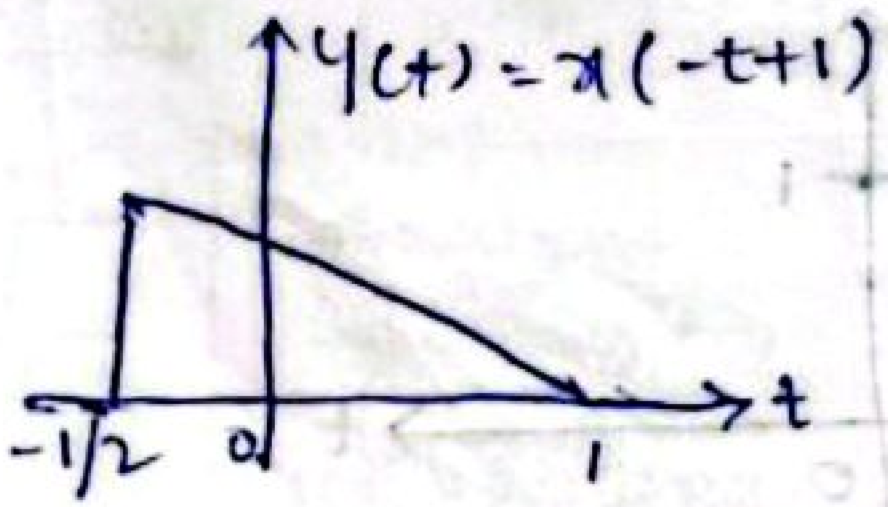
③



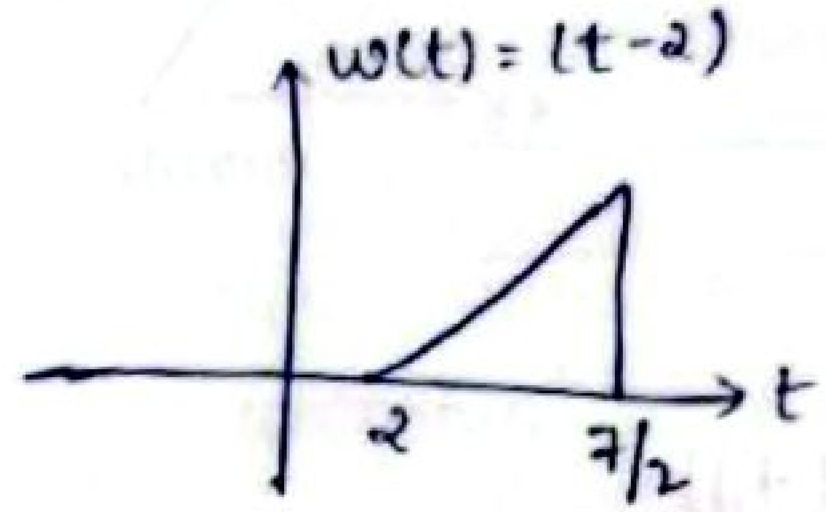
$x(-t+1)$



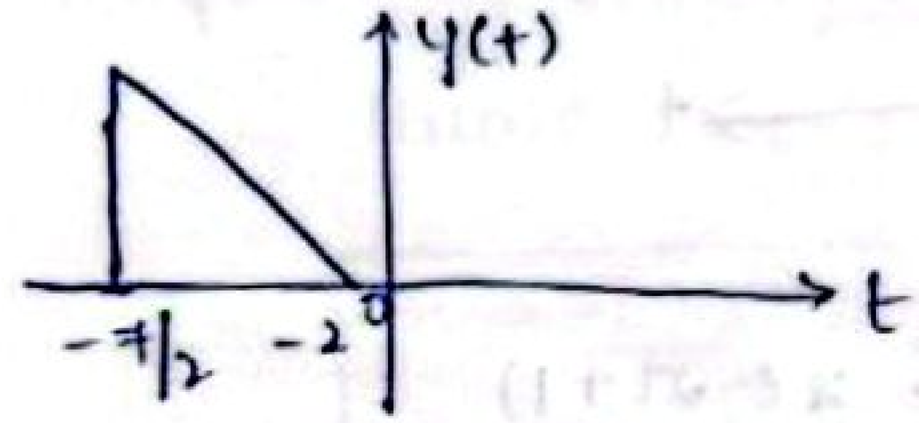
$y(t) = w(-t) = x(-t+1)$



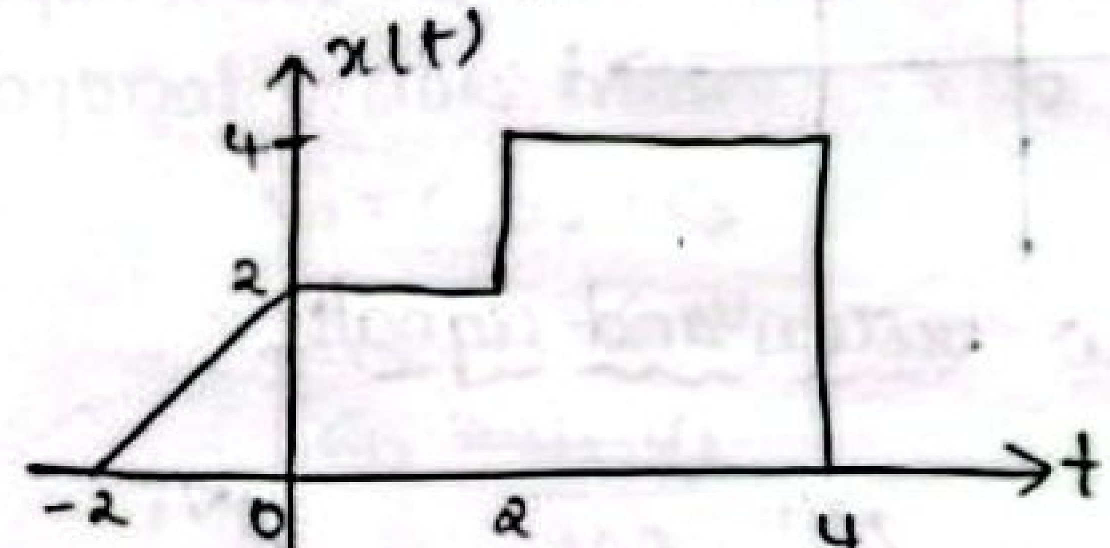
④ $x(-t-2)$



$y(t) = w(-t) = x(-t-2)$

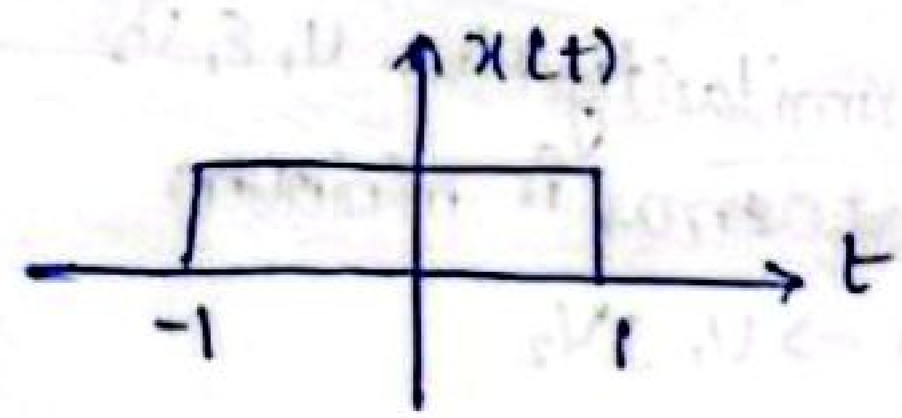


Q find, sketch the signal $x(4-t)$

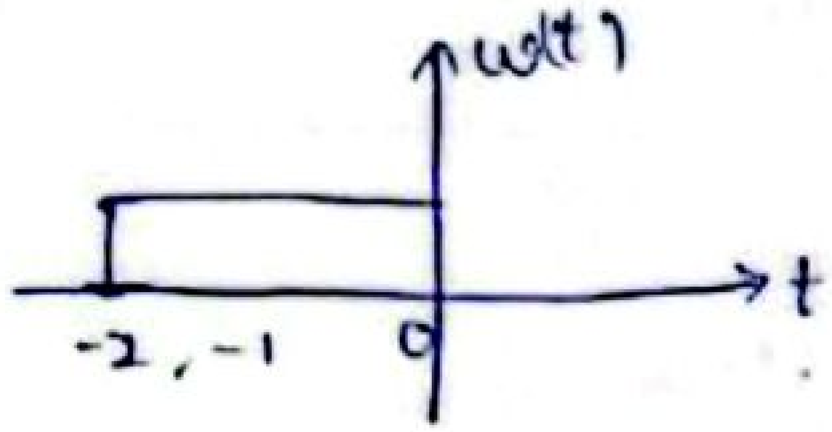


let $x(4-t) = x(-t+4)$

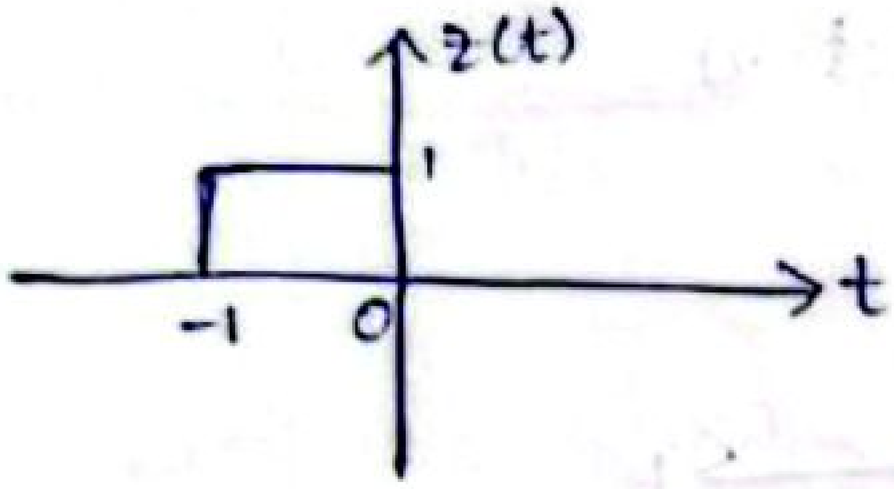
Q $x(t) = rect(t/2)$ sketch $y(t) = x(-2t+1)$



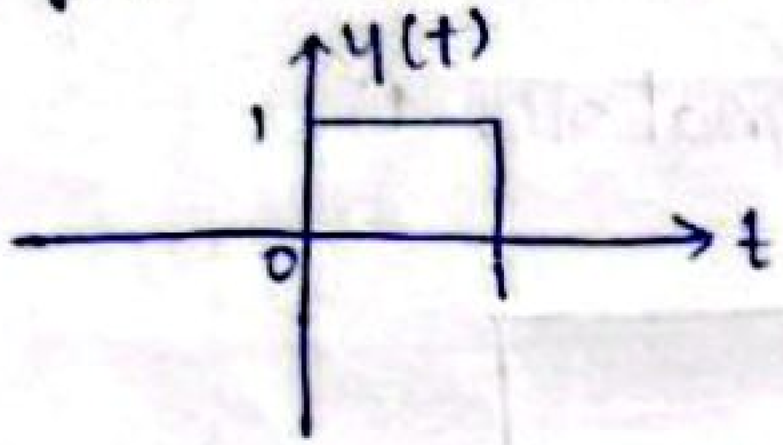
iii) $w(t) = x(t+1)$



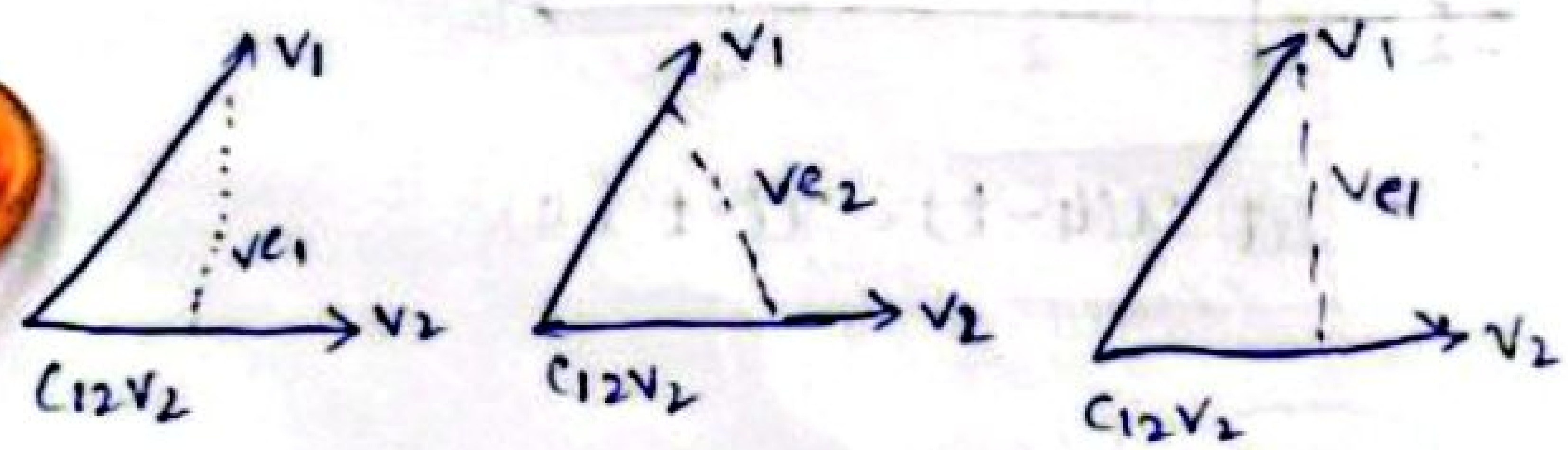
let $z(t) = w(2t) = x(2t+1)$



let $y(t) = z(-t) = x(-2t+1)$



The Anology b/w vectors and signals



$V_1 = C_{12}V_2 + V_{e1}$ $\quad V_1 = C_{12}V_2 + V_{e2}$ $\quad V_1 = C_1 V_2 + V_e$

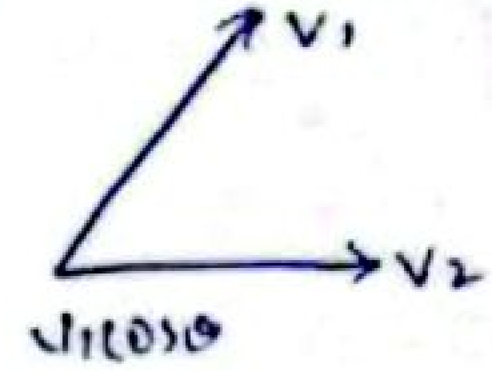
Error vector

$V_e = V_1 - C_{12}V_2$

$C_{12} \rightarrow$ The measure of similarity b/w V_1 & V_2

choose C_{12} such that error is minimum

$C_{12} = 0 \Rightarrow V_1 \perp V_2$



$V_1 \cdot V_2 = V_1 V_2 \cos\theta$

component of V_1 along V_2 is $V_1\cos\theta$

$V_1\cos\theta = \dfrac{V_1 \cdot V_2}{V_2}$

component of V_1 along V_2 is also $C_{12}V_2$

$V_1\cos\theta = C_{12}V_2 = \dfrac{V_1 \cdot V_2}{V_2}$

$$\boxed{C_{12} = \frac{V_1 \cdot V_2}{V_2^2} = \frac{V_1 \cdot V_2}{V_2 \cdot V_2}}$$

orthogonal vectors

orthogonal vectors means $V_1 \perp V_2$ and $C_{12} = 0$

$V_1 = C_{12}V_2 + C_e$

$V_e = V_1 - C_{12}V_2$ — ①

$C_{12} = \dfrac{V_1 \cdot V_2}{V_2^2}$

If we choose C_{12} like this

error is minimum

$\because V_e \cdot V_2 = 0$

from ①

$\underbrace{V_e \cdot V_2}$ $= V_1 \cdot V_2 - C_{12} V_2 \cdot V_2$

$0 = V_1 \cdot V_2 - C_{12} V_2 \cdot V_2$

$C_{12} = \dfrac{V_1 \cdot V_2}{V_2 \cdot V_2}$

⑤ $v_1 = 3i+3j+3k$

$v_2 = i+j+k$

$v_1 = c_{12}v_2 + v_e$

$$c_{12} = \frac{v_1 \cdot v_2}{v_2^2}$$

$v_1 \cdot v_2 = (3i+3j+3k)\cdot(i+j+k)$

$v_2^2 = v_2 \cdot v_2 = (i+j+k)\cdot(i+j+k)$

$$c_{12} = \frac{3+3+3}{3} = 3$$

$v_e = v_1 - c_{12}v_2$

$v_e = (3i+3j+3k) - 3(i+j+k)$

$v_e = 0$

⑥ $v_1 = 3i+3j+4k$

$v_2 = i+j+k$

find c_{12}

$v_1 = c_{12}v_2 + v_e$

$$c_{12} = \frac{v_1 \cdot v_2}{v_2^2}$$

$v_1 \cdot v_2 = (3i+3j+4k)\cdot(i+j+k)$

$v_2^2 = v_2 \cdot v_2 = (i+j+k)\cdot(i+j+k)$

$$c_{12} = \frac{3+3+4}{3} = \frac{10}{3}$$

$v_e = (3i+3j+4k) - \frac{10}{3}(i+j+k)$

$= (3-\frac{10}{3})i + (3-\frac{10}{3})j + (4-\frac{10}{3})k$

$= \frac{-1}{3}i - \frac{1}{3}j + \frac{2}{3}k$

$v_1 = \frac{10}{3}(i+j+k) + (-\frac{1}{3}i - \frac{1}{3}j + \frac{2}{3}k)$

$= 3i + 3j + 4k$

$v_e \cdot v_2 = 0 \Rightarrow (-\frac{1}{3}i - \frac{1}{3}j + \frac{2}{3}k)\cdot(i+j+k)$

$= -\frac{1}{3} - \frac{1}{3} + \frac{2}{3}$

$= 0$

- Signal approximation with respect to another signal

$x_1(t)$ & $x_2(t)$ be two signals ; $t_1 \le t \le t_2$

$x_1(t) = c_{12}x_2(t) + x_e(t)$

Error signal

$x_e(t) = x_1(t) - c_{12}x_2(t)$

find c_{12} such that the Error is minimum

Mean square value of the error

$$\varepsilon = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} x_e^2(t)\,dt \rightarrow \text{Mean square error criterion}$$

⇒ Mean square error criterion is chosen to minimize the Error.

⇒ To minimize this function we had to do diffuntion

$$\frac{d\varepsilon}{dc_{12}} = 0$$

Let $\varepsilon = \frac{1}{t_2-t_1}\int_{t_1}^{t_2} x_e^2(t)\,dt$

$$= \frac{1}{t_2-t_1}\int_{t_1}^{t_2}(x_1(t) - c_{12}x_2(t))^2\,dt$$

$$= \frac{1}{t_2-t_1}\int_{t_1}^{t_2}[x_1^2(t) + c_{12}^2x_2^2(t) - 2c_{12}x_1(t)x_2(t)]dt$$

$$\frac{dE}{dc_{12}} = \frac{\partial}{\partial c_{12}}\left[\frac{1}{t_1-t_2}\int_{t_1}^{t_2}(x_1^2(t)+c_{12}^2x_2^2(t)-2c_{12}x_1(t)x_2(t))\,dt\right]=0$$

$$\text{let } \frac{\partial E}{\partial c_{12}} = \frac{1}{t_1-t_2}\int_{t_1}^{t_2}\left[\frac{\partial}{\partial c_{12}}(x_1^2(t)) + \frac{\partial}{\partial c_{12}}c_{12}^2x_2^2(t) - 2\frac{\partial}{\partial c_{12}}c_{12}x_1(t)x_2(t)\right]dt$$

$$= \frac{1}{t_1-t_2}\int_{t_1}^{t_2}\left[0+2c_{12}x_2^2(t)-2x_1(t)x_2(t)\right]dt=0$$

$$\Rightarrow \frac{1}{t_1-t_2}\left[2c_{12}\int_{t_1}^{t_2}x_2^2(t)dt - 2\int_{t_1}^{t_2}x_1(t)x_2(t)dt\right]=0$$

$$\Rightarrow \int_{t_1}^{t_2}c_{12}x_2^2(t)dt = \int_{t_1}^{t_2}x_1(t)x_2(t)\,dt$$

$$\boxed{c_{12} = \frac{\int_{t_1}^{t_2}x_1(t)x_2(t)dt}{\int_{t_1}^{t_2}x_2^2(t)dt}}$$

→ This representation is called integral inner product b/w $x_1(t)$ & $x_2(t)$

Mean square error value

$$\varepsilon = \frac{1}{t_1-t_2}\int_{t_1}^{t_2}x_e^2(t)dt$$

$$= \frac{1}{t_1-t_2}\int_{t_1}^{t_2}(x_1^2(t)+x_2^2(t)c_{12}^2-2c_{12}x_1(t)x_2(t))\,dt$$

we know that

$$c_{12} = \frac{\int_{t_1}^{t_2}x_1(t)x_2(t)dt}{\int_{t_1}^{t_2}x_2^2(t)dt}$$

$$c_{12}\int_{t_1}^{t_2}x_2^2(t)dt = \int_{t_1}^{t_2}x_1(t)x_2(t)dt \quad \text{—①}$$

$$\therefore \frac{1}{t_1-t_2}\left[\int_{t_1}^{t_2}x_1^2(t)dt + c_{12}^2\int_{t_1}^{t_2}x_2^2(t)dt - 2c_{12}\int_{t_1}^{t_2}x_1(t)x_2(t)dt\right]$$

from ①

$$\frac{1}{t_1-t_2}\left[\int_{t_1}^{t_2}x_1^2(t)dt + c_{12}\int_{t_1}^{t_2}x_1(t)x_2(t)dt - 2c_{12}\int_{t_1}^{t_2}x_1(t)x_2(t)dt\right]$$

$$\varepsilon = \frac{1}{t_1-t_2}\left[\int_{t_1}^{t_2}x_1^2(t)dt - c_{12}\int_{t_1}^{t_2}x_1(t)x_2(t)dt\right]$$

(Or)

from ①

$$\frac{1}{t_1-t_2}\left[\int_{t_1}^{t_2}x_1^2(t)dt + c_{12}^2\int_{t_1}^{t_2}x_2^2(t)dt - 2c_{12}\cdot c_{12}\int_{t_1}^{t_2}x_2^2(t)dt\right]$$

from ①

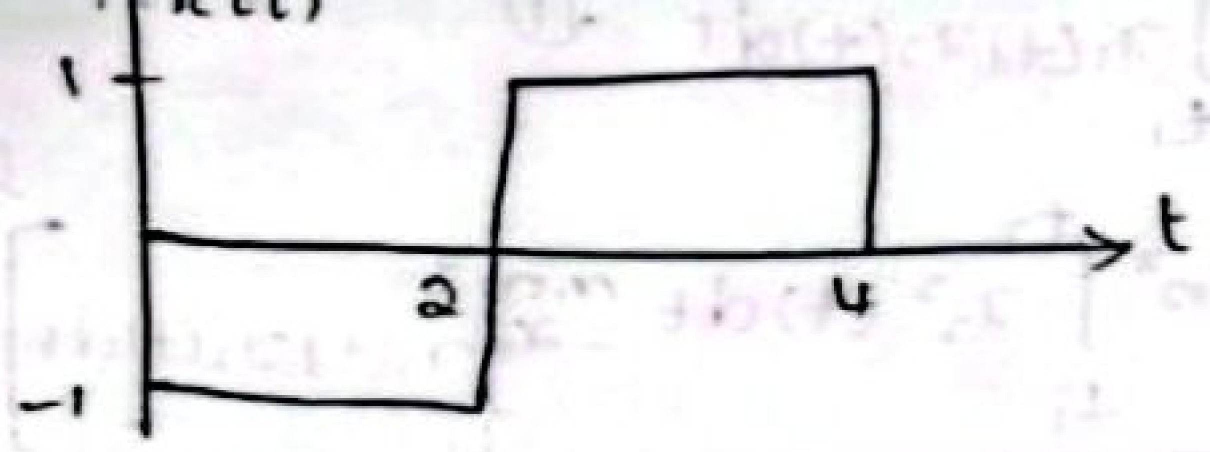
$$\boxed{\frac{1}{t_1-t_2}\left[\int_{t_1}^{t_2}x_1^2(t)dt - c_{12}^2\int_{t_1}^{t_2}x_2^2(t)dt\right]}$$

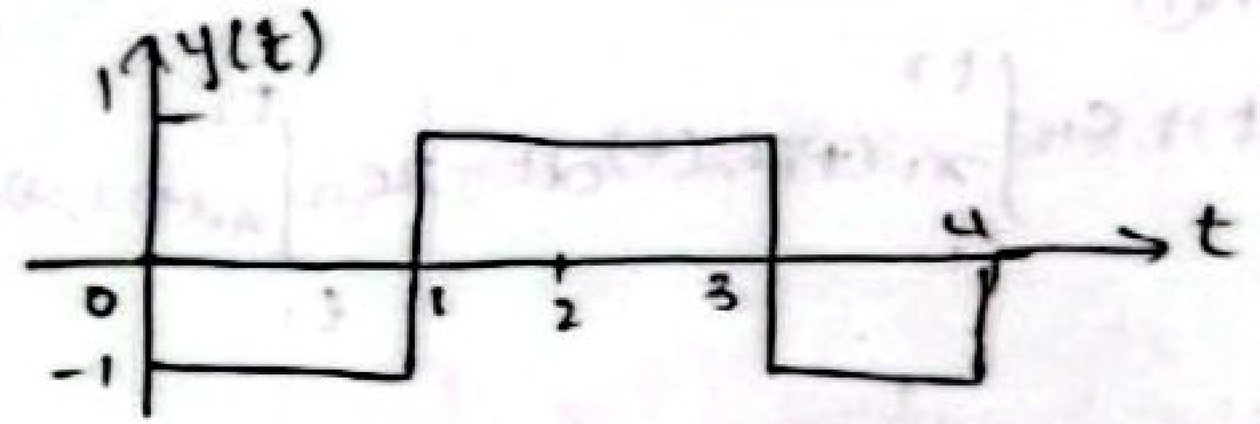
→ $c_{12}=0$ means, $x_1(t)$ & $x_2(t)$ are orthogonal

Orthogonal signal

$$c_{12}=0, \quad \therefore \int_{t_1}^{t_2}x_1(t)\cdot x_2(t)dt=0$$

Q) check whether given signals are orthogonal (or) not





$$\int_0^4 x(t)\,y(t)\,dt$$

$$= \int_0^1 (-1)(-1)\,dt + \int_1^2 (-1)(1)\,dt + \int_2^3 (1)(1)\,dt + \int_3^4 (1)(-1)\,dt$$

$$= \int_0^1 1\,dt - \int_1^2 dt + \int_2^3 dt - \int_3^4 dt$$

$$= (t)_0^1 - (t)_1^2 + (t)_2^3 - (t)_3^4$$

$$= (1-0) - (2-1) + (3-2) - (4-3)$$

$$= 1 - 1 + 1 - 1$$

$$= 0$$

∴ These two signals are orthogonal.

For complex signals:-

$$\int_{t_1}^{t_2} x_1^*(t)\,x_2(t)\,dt = \int_{t}^{t_2} x_1(t)\,x_2^*(t)\,dt = 0$$ (integral inner product)

Then we can say the two complex signals and $x_2(t)$ are orthogonal

(5) Prove that the following complex exponential signals are orthogonal over the interval $t_0, t_0+\frac{2\pi}{\omega_0}$

$x_1(t) = e^{jm\omega_0 t}$, $x_2(t) = e^{jn\omega_0 t}$

let $$\int_{t_1}^{t_2} x_1(t)\,x_2^*(t)\,dt = 0$$

$$\Rightarrow \int_{t_0}^{t_0+2\pi/\omega_0} e^{jm\omega_0 t} \cdot e^{-jn\omega_0 t}\,dt = 0$$

$$\Rightarrow \int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} e^{j(m-n)\omega_0 t}\,dt = 0$$

$$= \left[\frac{e^{j(m-n)\omega_0 t}}{j(m-n)\omega_0}\right]_{t_0}^{t_0+\frac{2\pi}{\omega_0}}$$

$$= \frac{1}{j(m-n)\omega_0}\left[e^{j(m-n)\omega_0\left(t_0+\frac{2\pi}{\omega_0}\right)} - e^{j(m-n)\omega_0 t_0}\right]$$

$$= \frac{1}{j(m-n)\omega_0}\left[e^{j(m-n)\omega_0 t_0}\left[e^{j(m-n)\omega_0\frac{2\pi}{\omega_0}} - 1\right]\right]$$

$$= \frac{1}{j(m-n)\omega_0}\left[e^{j(m-n)\omega_0 t_0}\left[e^{j(m-n)2\pi} - 1\right]\right]$$

$$= \cancel{tt}\; e^{j(m-n)2\pi} - 1$$

$$= \frac{1}{j(m-n)\omega_0} e^{j(m-n)\omega_0}[1-1]$$

$$= 0$$

Q) show that the following signals are orthogonal over the interval $\left(t_0, t_0+\frac{2\pi}{\omega_0}\right)$

① $\sin(n\omega_0 t)$ and $\sin(m\omega_0 t)$

where n,m are ~~constant~~ integers and

this two signals are orthogonal

② $\sin(n\omega_0 t)$ and $\cos(m\omega_0 t)$

③ $\cos(n\omega_0 t)$ and $\cos(m\omega_0 t)$

② let $x_1 = \sin(n\omega_0 t)$, $x_2 = \cos(m\omega_0 t)$

$$\int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} \sin(n\omega_0 t)\cos(m\omega_0 t)\,dt$$

multiply and divided with 2

$$\frac{1}{2}\int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} 2\sin(n\omega_0 t)\cos(m\omega_0 t)\,dt$$

$$\Rightarrow \frac{1}{2}\int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} \left(\sin((n+m)\omega_0 t) + \sin((n-m)\omega_0 t)\right)dt$$

$$\Rightarrow \frac{1}{2}\left[\left[\frac{-\cos(n+m)\omega_0 t}{(n+m)\omega_0}\right]_{t_0}^{t_0+\frac{2\pi}{\omega_0}} - \left[\frac{\cos(n-m)\omega_0 t}{(n-m)\omega_0}\right]_{t_0}^{t_0+\frac{2\pi}{\omega_0}}\right]$$

$$\Rightarrow -\frac{1}{2}\left[\left[\frac{\cos(n+m)\omega_0\left(t_0-\frac{2\pi}{\omega_0}\right) - \cos(n+m)\omega_0 t_0}{(n+m)\omega_0}\right] + \left[\frac{\cos(n-m)\omega_0\left(t_0+\frac{2\pi}{\omega_0}\right) - \cos(n-m)\omega_0 t_0}{(n-m)\omega_0}\right]\right]$$

$$= 0$$

∴ The signals are orthogonal

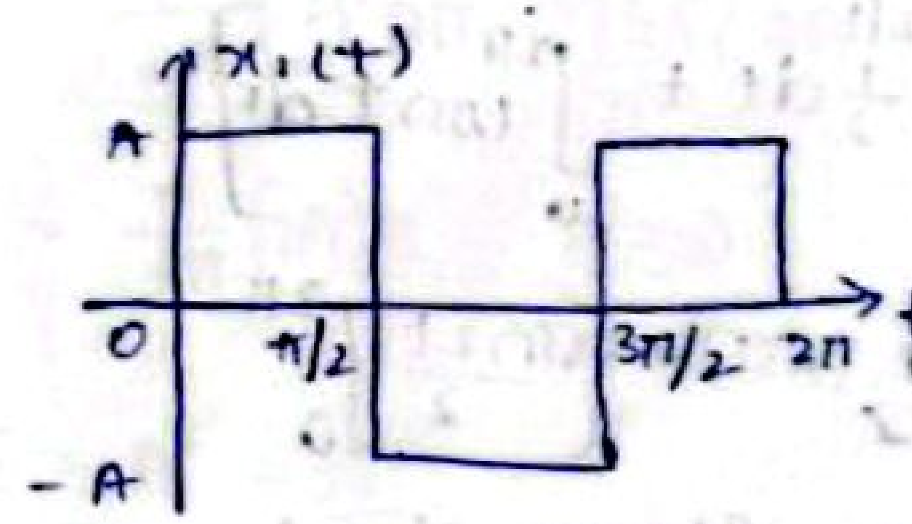
*** Q) A rectangle function is defined as

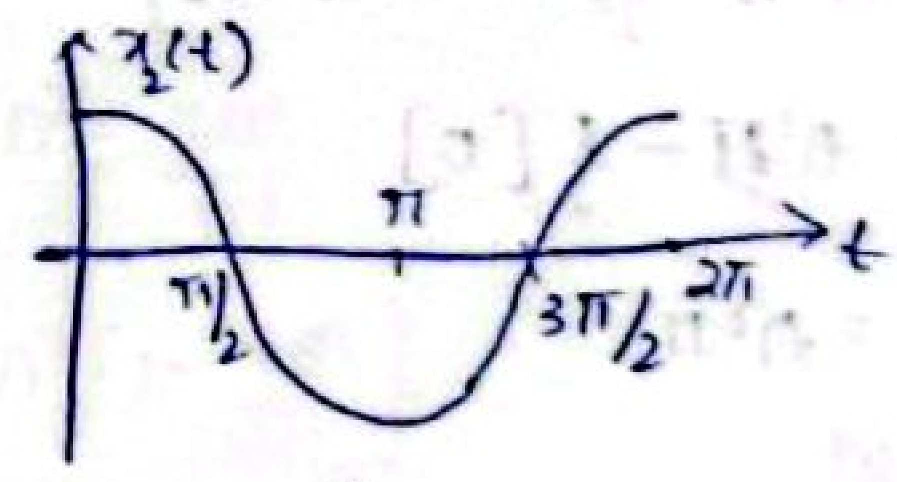
$$x(t) = \begin{cases} A & \text{for } 0 < t < \pi/2 \\ -A & \text{for } \pi/2 < t < 3\pi/2 \\ A & \text{for } 3\pi/2 < t < 2\pi \end{cases}$$

Approximate the above function by $A\cos(t)$ over the interval $(0, 2\pi)$ such that the mean square Error is minimum

let $x_1(t) = x(t)$ and $x_2(t) = A\cos(t)$

$$x_1(t) \simeq C_{12}x_2(t)$$





$$C_{12} = \frac{\int_{t_1}^{t_2} x_1(t)x_2(t)\,dt}{\int_{t_1}^{t_2} x_2^2(t)\,dt}$$

$t_1 = 0$ and $t_2 = 2\pi$

$$\because \int_0^{2\pi} x_1(t)\, x_2(t)\, dt = \int_0^{\pi/2} A\cdot A\cos(t)\, dt + \int_{\pi/2}^{3\pi/2} (-A)(A\cos t)\, dt + \int_{3\pi/2}^{2\pi} A\cdot A\cos t\, dt$$

$$= A^2(\sin t)_0^{\pi/2} - A^2(\sin t)_{\pi/2}^{3\pi/2} + A^2(\sin t)_{3\pi/2}^{2\pi}$$

$$= A^2\left[\sin\tfrac{\pi}{2} - \sin 0\right] - A^2\left[\sin\tfrac{3\pi}{2} - \sin\tfrac{\pi}{2}\right] + A^2[0+1]$$

$$= A^2[1-0] - A^2(-1-1) + A^2(0+1)]$$

$$= 4A^2$$

$$\int_0^{2\pi} x_2^2\, dt = \int_0^{2\pi} A^2\cos^2 t\, dt$$

$$= A^2\int_0^{2\pi} \frac{1+\cos 2t}{2}\, dt$$

$$= A^2\left[\int_0^{2\pi}\frac{1}{2}\, dt + \int_0^{2\pi}\cos 2t\, dt\right]$$

$$= A^2\,\frac{1}{2}(2\pi) + \left[\frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$= A^2\pi - \frac{1}{2}[\sin 4\pi - \sin 0]$$

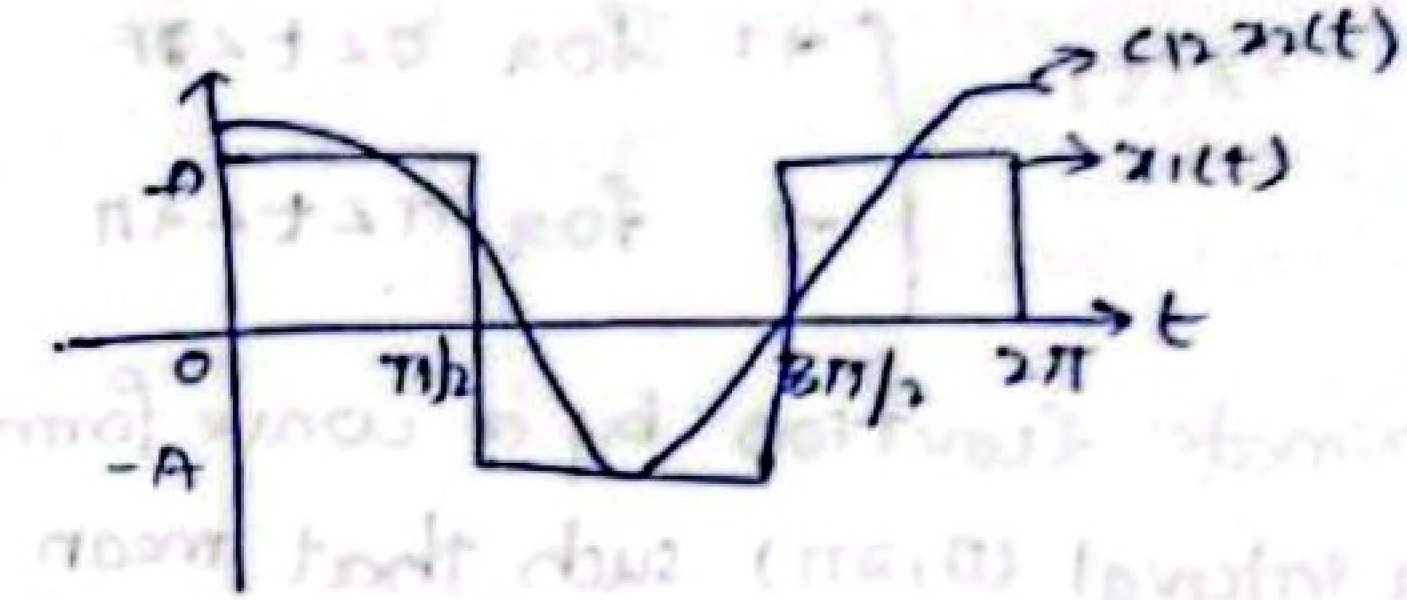
$$= A^2\pi - \frac{1}{2}[0]$$

$$= A^2\pi$$

$$C_{12} = \frac{4\cancel{A^2}}{\cancel{A^2}\pi}$$

$$C_{12} = \frac{4}{\pi}$$

$$x_1(t) \simeq \frac{4}{\pi}A\cos(t)$$



Mean square error value

$$\varepsilon = \frac{1}{t_2-t_1}\left[\int_{t_1}^{t_2} x_1^2(t)\, dt - C_{12}^2\int_{t_1}^{t_2} x_2^2(t)\, dt\right]$$

$$\varepsilon = \frac{1}{2\pi}\left[\int_0^{2\pi} A^2\, dt - \left(\frac{4}{\pi}\right)^2\int_0^{2\pi} A^2\cos^2(t)\, dt\right]$$

$$\varepsilon = \frac{1}{2\pi}\left[A^2 2\pi - \frac{16}{\pi^2}\cdot A^2\int_0^{2\pi}\left(\frac{1+\cos 2t}{2}\right) dt\right]$$

$$\varepsilon = \frac{1}{2\pi}\left[A^2 2\pi - \frac{16A^2}{\pi^2}(2\pi - 0)\right]$$

$$\varepsilon = \frac{1}{2\pi}\left[2A^2\pi - \frac{A^2 32}{2\pi}\right]$$

$$\varepsilon = A^2 - \frac{32A^2}{4\pi^2}$$

$$\varepsilon = A^2 - \frac{8A^2}{\pi^2}$$

$$\varepsilon = A^2\left[1 - \frac{8}{\pi^2}\right]$$

$$\varepsilon \approx A^2(0.2)$$

let $A = 1$

$\varepsilon = 20\%$

⑥ A rectangular function

$$x(t) = \begin{cases} +1 & \text{for } 0 < t < \pi \\ -1 & \text{for } \pi < t < 2\pi \end{cases}$$

Approximate function by a wave form $\sin t$ over a interval $(0, 2\pi)$ such that mean square error is minimum n-dimensional vector space

→ In n-dimensional vector space $\{x_1, x_2, x_3 \cdots x_n\}$ This are orthogonal (or) orthonormal vector, basis vector and complete set

→ A vector which is belong to n-dimensional vector can be represented by

$$F = c_1x_1 + c_2x_2 + c_3x_3 + \cdots + c_rx_r + \cdots + c_nx_n$$

→ $x_m \cdot x_n = \begin{cases} 0 & \text{if } m \neq n \\ k_m & \text{if } m = n \end{cases}$

orthogonalisation

→ if $k_m = 1$ the vectors are orthonormal

→ $F \cdot x_r = c_1 \cdot \underbrace{x_1 \cdot x_r}_{0} + c_2 \underbrace{x_2 x_r}_{0} + \cdots + c_r x_r \cdot x_r + \cdots + c_n \underbrace{x_n x_r}_{0}$

Because both are orthogonal

→ $c_r = \dfrac{F \cdot x_r}{x_r \cdot x_r}$ for $r = 1, 2, \ldots, n$

orthonormal vector

$x_r \cdot x_r = 1$

$\therefore c_r = F \cdot x_r$

*Approximation of a function using set of n-mutually orthogonal functions.

Let us consider $\{g_1(t), g_2(t) \; g_r(t) \; g_n(t)\}$; (t_1, t_2)

The set of functions are mutually orthogonal

$$\int_{t_1}^{t_2} g_j(t) \cdot g_k(t)\,dt = \begin{cases} 0 & \text{if } j \neq k \\ k_j & \text{if } j = k \end{cases}$$

components:

Let $(c_1, c_2 \cdots c_r \cdots c_n)$

$x(t) \approx c_1 g_1(t) + c_2 g_2(t) + c_3 g_3(t) + \cdots + c_n g_n(t)$

$$x(t) \approx \sum_{r=1}^{n} c_r g_r(t)$$

Error signal

$$x_e(t) = x(t) - \sum_{r=1}^{n} c_r g_r(t)$$

compute c_1 to c_n such that mean square vale is minimum.

$$\varepsilon = \frac{1}{t_1 - t_2} \int_{t_1}^{t_2} x_e^2(t)\,dt$$

$$= \frac{1}{t_1 - t_2} \int_{t_1}^{t_2} \left[x(t) - \sum_{r=1}^{n} c_r g_r(t)\right]^2 dt$$

$$\frac{\partial \varepsilon}{\partial c_1} = \frac{\partial \varepsilon}{\partial c_2} = \cdots = \frac{\partial \varepsilon}{\partial c_j} = \cdots = \frac{\partial \varepsilon}{\partial c_n} = 0$$

Let $\dfrac{\partial \varepsilon}{\partial c_j} = 0$, for $j = 1, 2 \cdots n$

$$\frac{\partial}{\partial c_j} \cdot \left[\frac{1}{t_1 - t_2} \int_{t_1}^{t_2} \left(x(t) - \sum_{r=1}^{n} c_r g_r(t)\right)^2 dt\right] = 0$$

$$\frac{\partial}{\partial c_j} \left\{ \frac{1}{t_1 - t_2} \int_{t_1}^{t_2} x^2(t) + \sum_{r=1}^{n} c_r^2 g_r^2(t) - 2x(t) \sum_{r=1}^{n} c_r g_r(t) \right\} = 0$$

~~let $x \neq 2$~~

$\Rightarrow \displaystyle\int_{t_1}^{t_2} \frac{\partial}{\partial c_j} (x^2(t)\,dt + \int_{t_1}^{t_2} [c_1^2 g_1^2(t) + c_2^2 g_2^2(t) + \cdots + c_j^2 g_j^2(t) + \cdots$

$\cdots + c_n^2 g_n^2(t)]\,dt = 2\displaystyle\int_{t_1}^{t_2} x(t) \frac{\partial}{\partial c_j} [c_1 g_1(t) + c_2 g_2(t) \cdots + c_j g_j(t) \cdots$

$\cdots + c_n g_n(t)]\,dt$

$$\Rightarrow 0 + \int_{t_1}^{t_2} \left[0 + 0 + \cdots + 2c_j g_j^2(t) + \cdots 0 \right] dt + \int_{t_1}^{t_2} x(t) \left[0 + 0 + \cdots + g_j(t) + \cdots 0 \right] dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2} \cancel{2} c_j g_j^2(t) dt - \cancel{2} \int_{t_1}^{t_2} x(t)\, g_j(t)\, dt = 0$$

$$c_j \int_{t_1}^{t_2} g_j^2(t) dt = \int_{t_1}^{t_2} x(t) g_j(t)\, dt$$

$$c_j = \frac{\int_{t_1}^{t_2} x(t) \cdot g_j(t) dt}{\int_{t_1}^{t_2} g_j^2(t)\, dt} \quad ; j = 1, 2, \cdots, n$$

Mean scale error value (MSE)

$$\varepsilon = \frac{1}{t_2 - t_1} \left[\int_{t_1}^{t_2} x_e^2(t) \cdot dt \right]$$

$$= \frac{1}{t_2 - t_1} \left[\int_{t_1}^{t_2} \left[x(t) - \sum_{r=1}^{n} c_r g_r(t) \right]^2 dt \right]$$

$$= \frac{1}{t_2 - t_1} \left[\int_{t_1}^{t_2} \left[x^2(t) + \sum_{r=1}^{n} c_r^2 g_r^2(t) - 2x(t) \sum_{r=1}^{n} c_r g_r(t) \right] dt \right]$$

$$= \frac{1}{t_2 - t_1} \left[\int_{t_1}^{t_2} x^2(t) + \sum_{r=1}^{n} c_r^2 \int_{t_1}^{t_2} g_r^2(t) \cdot dt - 2 \sum_{r=1}^{n} c_r \int_{t_1}^{t_2} x(t) \cdot g_r(t)\, dt \right]$$

$$c_r = \frac{\int_{t_1}^{t_2} x(t) \cdot g_r(t) dt}{\int_{t_1}^{t_2} g_r(t)\, dt} \quad ; \text{for } r = 1, 2, \cdots n$$

$$\therefore \int_{t_1}^{t_2} x(t) \cdot g_r(t) dt = c_r \int_{t_1}^{t_2} g_r(t) dt$$

$$\varepsilon = \frac{1}{t_1 - t_2} \left[\int_{t_1} x^2(t) + \sum_{r=1} c_r^2 \int_{t_1} g_r^2(t) dt - 2 \sum_{r=1} c_r \int_{t_1} g_r(t) dt \right]$$

$$\boxed{\varepsilon = \frac{1}{t_2 - t_1} \left[\int_{t_1}^{t_2} x^2(t) - \sum_{r=1}^{n} c_r^2 E_r \right]}$$

where $E_r = \int_{t_1}^{t_2} g_r^2(t) \cdot dt$

(a) A rectangular function $x(t) = \begin{cases} 1 & \text{for } 0 < t < \pi \\ -1 & \text{for } \pi < t < 2\pi \end{cases}$

is (a) approximate by the function $\sin(t)$ over the interval $(0, 2\pi)$

(b) what happens of your approximate the above function using a set of sinusoidal functions over the interval $(0, 2\pi)$

(a) $x(t) = c_{12} x_2(t)$

$x_1(t) = x(t)$

$x_2(t) = \sin(t)$

$\therefore x(t) = c_{12} \sin(t)$

$$c_{12} = \frac{\int_{t_1}^{t_2} x_1(t) x_2(t) dt}{\int_{t_1}^{t_2} x_2^2(t) dt} = \frac{\int_0^{2\pi} x(t) \sin(t)\, dt}{\int_0^{2\pi} \sin^2(t) dt}$$

$$= \frac{4}{\pi}$$

$$\varepsilon_1 = \frac{1}{t_2 - t_1} \left[\int_{t_1} x_1^2(t) - (c_{12})^2 \int_{t_1}^{t_2} x_2^2(t)\, dt \right]$$

$$\varepsilon \approx 0.188$$

(b) $g_1(t) = \sin(2t)$

$\{g_1(t), g_2(t) \cdots \cdots g_n(t)\}$

$\{\sin(t), \sin(2t) \cdots \cdots \sin(nt)\}$

$x(t) \simeq c_1\sin(t) + c_2\sin(2t) + \cdots + c_n\sin(nt)$

$$x(t) \simeq \sum_{r=1}^{n} c_r \sin(rt)$$

$$c_r = \frac{\int_{t_1}^{t_2} x(t)\cdot g_r(t)\,dt}{\int_{t_1}^{t_2} g_r^2(t)\cdot dt} = \frac{\int_0^{2\pi} x(t)\sin(rt)\,dt}{\int_0^{2\pi} \sin^2(rt)\,dt}$$

$$\int_0^{2\pi} x(t)\sin(rt)\,dt = \int_0^{\pi} (+1)\sin(rt)\,dt + \int_{\pi}^{2\pi} (-1)\sin(rt)\,dt$$

$$\int_0^{\pi} \sin(rt)\cdot dt = -\left[\frac{\cos(rt)}{r}\right]_0^{\pi}$$

$$= -\frac{1}{r}\left[\cos r\pi - \cos 0\right]$$

$$= \frac{+1-(-1)^r}{r}$$

$$-\int_{\pi}^{2\pi} \sin(rt)\,dt = \left[\frac{\cos(rt)}{r}\right]_{\pi}^{2\pi} = \frac{1}{r}\left[\cos(2\pi r) - \cos(\pi r)\right]$$

$$= \frac{1}{r}\left[1-(-1)^r\right]$$

$$\int_0^{2\pi} x(t)\sin(rt)\,dt = \frac{2}{r}\left[1-(-1)^r\right]$$

$$\int_0^{2\pi} \sin^2(rt)\cdot dt = \pi$$

$$c_r = \frac{2}{r\pi}\left[1-(-1)^r\right]$$

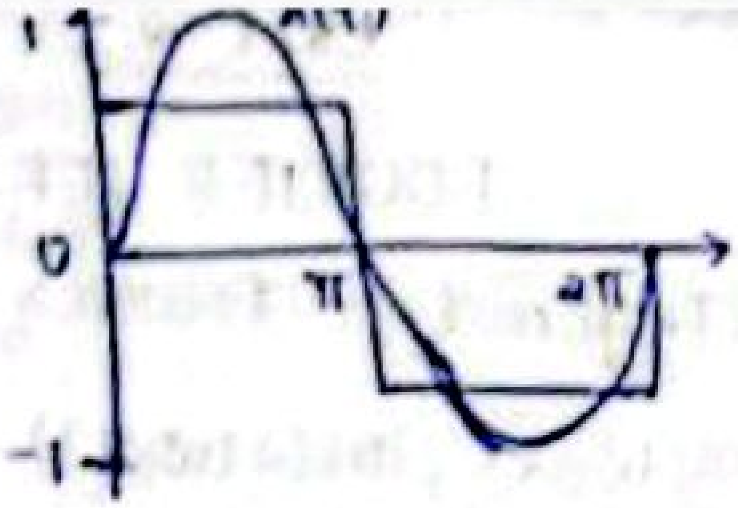
$$c_r = \begin{cases} 0 & \text{if } r \text{ is even} \\ \frac{4}{r\pi} & \text{if } r \text{ is odd} \end{cases}$$

case ①

$n = 1,\ c_1 = \frac{4}{\pi}$

$x(t) = \frac{4}{\pi}\sin(t)$

$\varepsilon_1 \simeq 0.18$

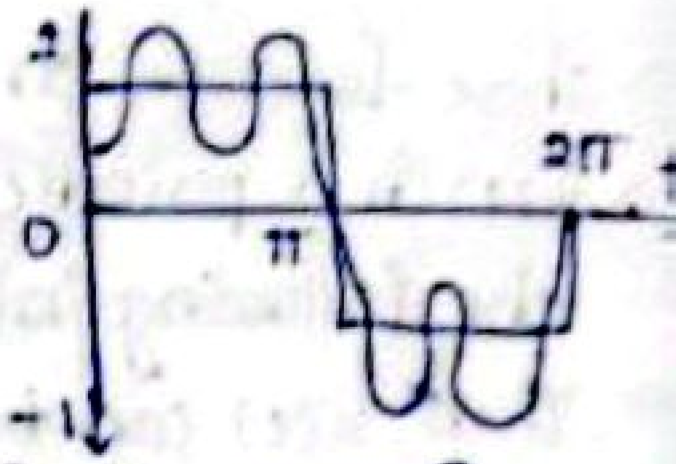


case ②

$n = 3$

$x(t) = c_1\sin(t) + c_2\sin(2t) + c_3\sin(3t)$

$x(t) = \frac{4}{\pi}\sin(t) + 0 + \frac{4}{\pi}\sin(3t)$



$$\varepsilon_2 = \frac{1}{t_2 - t_1}\left[\int_{t_1}^{t_2} x^2(t)\,dt - (c_1^2E_1 + c_2^2E_2 + c_3^2E_3) - \cos(\pi 2)\right]$$

$c_1 = \frac{4}{\pi},\ c_3 = \frac{4}{3\pi},\ c_2 = 0$

$E_2 \simeq 0.099$

case ③

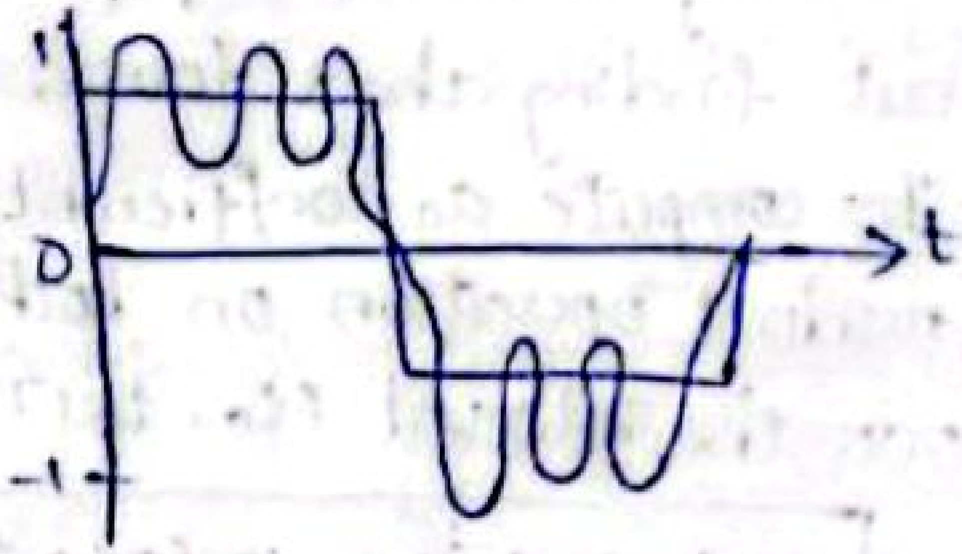
$n = 5$

$x(t) = c_1\sin(t) + c_2\sin(2t) + c_3\sin(3t) + c_4\sin(4t) + c_5\sin(5t)$

$x(t) = \frac{4}{\pi}\sin(t) + 0 + \frac{4}{3\pi}\sin(3t) + 0 + \frac{4}{5\pi}\sin(5t)$

$\varepsilon_3 \simeq 0.66$

as n goes on increases approximation is better and MSE decreases



Unit-2

FOURIER SERIES → used to locate frequency

TFS: (Trigometric fourier series)

$\{1, \cos(\omega_0 t), \cos(2(\omega_0 t)) \ldots\ldots \cos(n(\omega_0 t)) \ldots\ldots,$
$\sin(\omega_0 t), \sin(2\omega_0 t) \ldots\ldots \sin(n\omega_0 t)\}$

=> These function are called Basis functions for TFS

=> $x(t)$ is a periodic signal with a period 'T' & also satisfies the following 'Dirichlet's condition'.

→ Then $x(t)$ can be represented as the linear combination of the above basis functions.

Dirichlet's conditions:

① A function must have single valued function.

② A function must have finite no of maxima's & minima's

③ A function must have finite no of discontinuities

④ The signal must be a absolutely integrable signals

i.e. $\int_{-T/2}^{T/2} x(t)dt < \infty$

Finding down TFS of periodic signal $x(t)$ is nothing but finding their fourier coefficient i.e, a_0, a_n, b_n

→ To compute a_0 coefficient perform integral inner product operation on both sides of Eq ① with 1 over the interval (t_0, t_0+T)

$$\boxed{x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)} \quad -(1)$$

$$\int_{t_0}^{t_0+T} x(t)\cdot 1 dt = \int_{t_0}^{t_0+T} a_0(1)\cdot dt + \sum_{n=1}^{\infty}\int_{t_0}^{t_0+T}(1) a_n \cos(n\omega_0 t)dt + \sum_{n=1}^{\infty}\int_{t_0}^{t_0+T}(1) b_n \sin(n\omega_0 t)dt$$

As 1 and cos, 1 and sin are orthogonal to each other

Hence, integral 2 and 3 becomes 0

$$\int_{t_0}^{t_0+T} x(t)\cdot 1dt = a_0(t)\Big|_{t_0}^{t_0+T} + 0 + 0$$

$$\int_{t_0}^{t_0+T} x(t)\cdot dt = a_0 T$$

$$\boxed{a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)dt}$$ => Average value (or) DC component

* To compute a_n, b_n.

① $\int_{t_0}^{t_0+T} \cos(m\omega_0 t)\cdot\cos(m\omega_0 t)dt = \begin{cases} 0 & \text{if } m \neq n \\ T/2 & \text{if } m = n \end{cases}$

② $\int_{t_0}^{t_0+T} \sin(n\omega_0 t)\sin(m\omega_0 t)dt = \begin{cases} 0 & \text{if } m \neq n \\ T/2 & \text{if } m = n \end{cases}$

③ $\int_{t_0}^{t_0+T} \sin(n\omega_0 t)\cos(m\omega_0 t)dt = 0$

→ To compute a_n, perform integral inner product on both sides with $\cos(m\omega_0 t)$ of Eq ①

$$\int_{t_0}^{t_0+T} x(t)\cdot\cos(m\omega_0 t)dt - a_0\int_{t_0}^{t_0+T}\cos(m\omega_0 t)dt + \sum_{n=1}^{\infty} a_n \int_{t_0}^{t_0+T}\cos(n\omega_0 t)\cdot\cos(m\omega_0 t)dt + \sum_{n=1}^{\infty} b_n \int_{t_0}^{t_0+T}\sin(n\omega_0 t)\cos(m\omega_0 t)dt$$

$$\Rightarrow \int_{t_0}^{t_0+T} x(t)\cdot\cos(m\omega_0 t)dt = 0 + a_m\frac{T}{2} + 0$$

$$a_m = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cdot\cos(m\omega_0 t)\,dt$$

$m \to n$

$$\boxed{a_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cdot\cos(n\omega_0 t)\,dt} \quad \text{for, } n=1$$

→ To compute b_n: perform integral inner product operation on Both sides with $\sin(m\omega_0 t)\cdot$ over the interval (t_0, t_0+T)

$$\boxed{b_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cdot\sin(n\omega_0 t)\,dt}$$

→ $T \to$ FP (fundamental period)

$\omega_0 = \frac{2\pi}{T}$ r/sec

→ Fundamental frequency (or) first harmonic frequency

$\omega_0 = 2\pi f_0$

$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{T}$ Hz

→ 2nd harmonic frequency is $2\omega_0$ (or) $2f_0$

→ nth harmonic frequency is $n\omega_0$ (or) nf_0

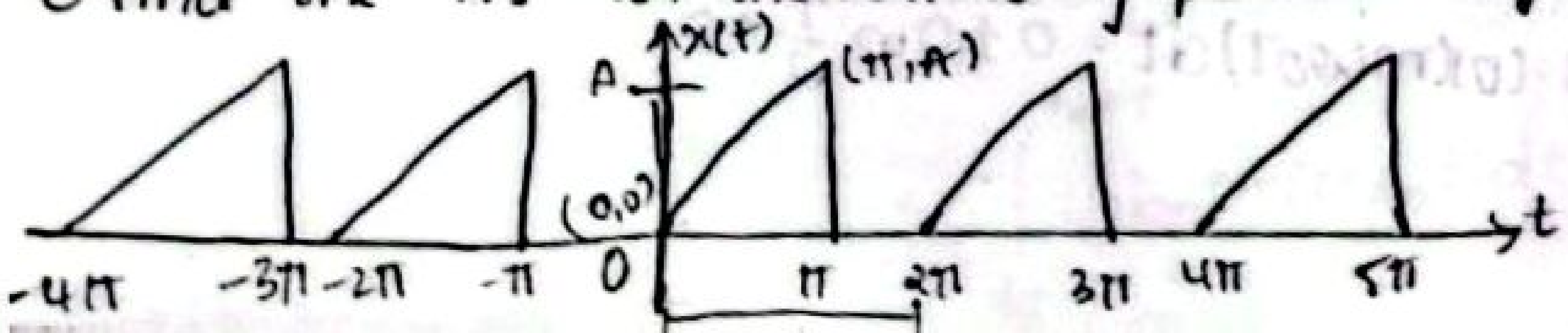
→ $x(t) = a_0 + \sum_{n=1}^{\infty} a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t)$

$a_1\cos(\omega_0 t) + b_1\sin(\omega_0 t) \to$ 1st harmonic component

$a_2\cos(2\omega_0 t) + b_2\sin(2\omega_0 t) \to$ 2nd

$a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t) \to$ nth

① Find the TFS for the following periodic signal



fundamental period

$T = 2\pi$

$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi}$

$= 1$ r/sec

$y = mx + c$

$x(t) = \left(\frac{A-0}{\pi-0}\right)t + c$

$x(A) = \frac{A}{\pi}t + c$

at $t = 0$

$x(0) = 0 + c$

$\boxed{c = 0}$

→ $a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\,dt$

let $t_0 = 0$

$a_0 = \frac{1}{T}\int_0^T x(t)\,dt = \frac{1}{2\pi}\int_0^{2\pi} x(t)\,dt$ $\quad x(t) = \begin{cases} \frac{A}{\pi}t \\ 0 \end{cases}$

$= \frac{1}{2\pi}\left[\int_0^{\pi}\frac{A}{\pi}t\,dt + \int_{\pi}^{2\pi} 0\,dt\right]$

$= \frac{A}{2\pi^2}\left[\frac{t^2}{2}\right]_0^{\pi}$

$= \frac{A}{4}$

→ $a_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cdot\cos(n\omega_0 t)\,dt$

$t_0 = 0$

$T = 2\pi$

$\omega_0 = 1$

$a_n = \frac{2}{2\pi}\int_0^{2\pi} x(t)\cdot\cos(n\omega_0 t)\,dt$

$= \frac{1}{\pi}\left[\int_0^{\pi}\frac{At}{\pi}\cdot\cos(n\omega_0 t)\,dt + \int_{\pi}^{2\pi} 0\cdot\cos(\omega_0 t)\,dt\right]$

$= \frac{A}{\pi^2}\left[\int_0^{\pi} t\cdot\cos(nt)\,dt\right]$

$= \frac{A}{\pi^2}\left[\left(\frac{t\sin(nt)}{n}\right)_0^{\pi} - \int_0^{\pi}\frac{\sin(nt)}{n}(1)\,dt\right]$

$$= \frac{A}{\pi^2}\left[\frac{\pi \sin(n\pi - 0)}{n} + \left(\frac{\cos(nt)}{n^2}\right)_0^{\pi}\right]$$

$$= \frac{A}{\pi^2}\left[0 + \frac{\cos(n\pi) - \cos(0)}{n^2}\right]$$

$$a_n = \frac{A}{\pi^2 n^2}\left[(-1)^n - 1\right], \text{ for } n = 1, 2, \dots \infty$$

$$a_n = \begin{cases} 0 & ; n \text{ is even} \\ \frac{-2A}{\pi^2 n^2} & ; n \text{ is odd} \end{cases}$$

$$\rightarrow b_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\sin(n\omega_0 t)\,dt$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{A}{\pi} t \sin(nt)\,dt$$

$$= \frac{A}{\pi^2}\left[\left(\frac{(-t)\cos(nt)}{n}\right)_0^{\pi} - \int_0^{\pi} \frac{-\cos(nt)}{n}\cdot 1\,dt\right]$$

$$= \frac{A}{\pi^2}\left[\frac{-\pi\cos(n\pi) - 0}{n} + \left(\frac{\sin(n\pi)}{n^2}\right)_0^{\pi}\right]$$

$$b_n = -\frac{A}{n\pi}(-1)^n$$

$$b_n = \begin{cases} \frac{-A}{n\pi} & \forall\, n \text{ is even} \\ \frac{A}{n\pi} & \forall\, n \text{ is odd} \end{cases}$$

TFS,

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n\cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

$$= \frac{A}{4} + \left(\frac{-2A}{\pi^2}\cos(t)\right) + 0 + \left(\frac{-2A}{4\pi^2}\cos(2t) + 0 + \dots\right.$$

$$\dots + \frac{A}{4}\sin(t) - \frac{A}{2\pi}\sin(2t) + \frac{A}{3\pi}\sin(3t) - \frac{A}{4\pi}\sin(4t) + \dots$$

Cosine TFS:-

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t)$$

$$= a_0 + \sum_{n=1}^{\infty}\sqrt{a_n^2 + b_n^2}\left[\frac{a_n}{\sqrt{a_n^2+b_n^2}}\cos(n\omega_0 t) + \frac{b_n}{\sqrt{a_n^2+b_n^2}}\sin(n\omega_0 t)\right]$$

Let $\cos(\theta_n) = \frac{a_n}{\sqrt{a_n^2+b_n^2}}$, $\sin(\theta_n) = \frac{-b_n}{\sqrt{a_n^2+b_n^2}}$

$C_0 = a_n$

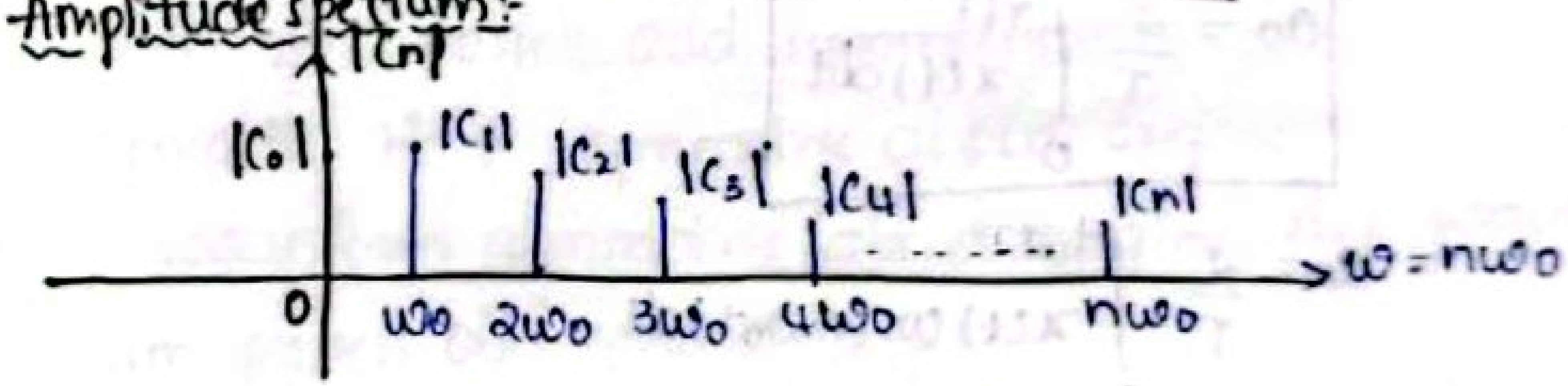
$C_n = \sqrt{a_n^2 + b_n^2}$

$\tan(\theta_n) = \frac{-b_n}{a_n} \Rightarrow \theta_n = -\tan^{-1}\left(\frac{b_n}{a_n}\right)$

$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n(\cos\theta_n \cos(n\omega_0 t) - \sin(\theta_n)\sin(n\omega_0 t))$$

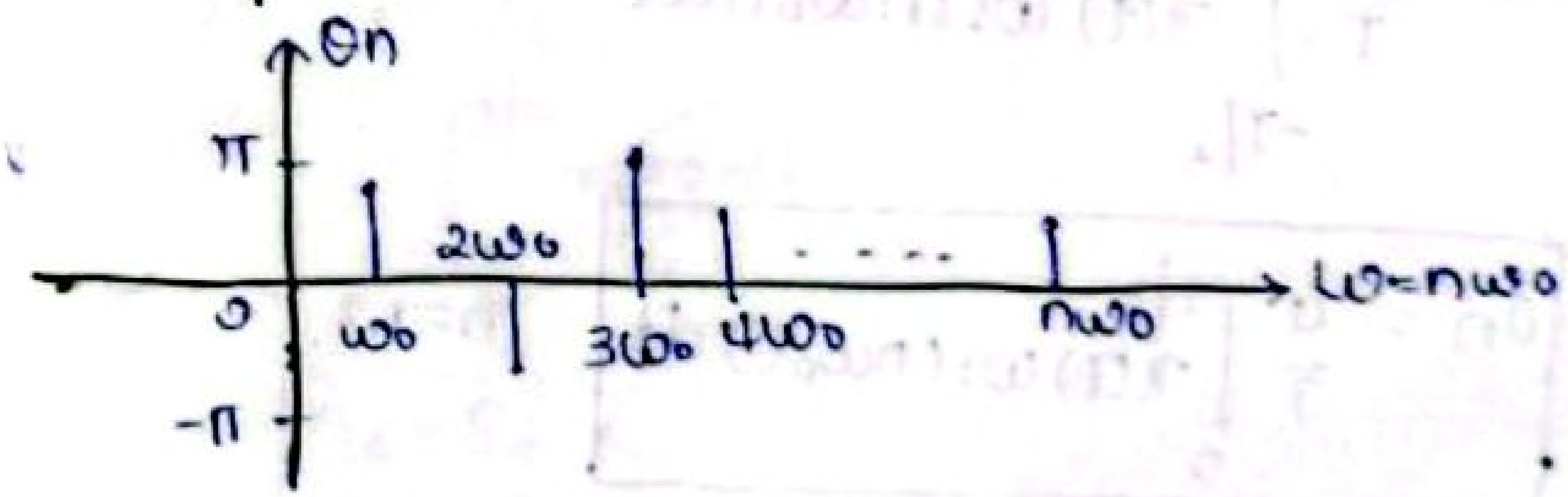
$$\boxed{x(t) = C_0 + \sum_{n=1}^{\infty} C_n\cos(n\omega_0 t + \theta_n)}$$

Amplitude spetrum:-



This plot says that how strong frequency's are.
Amplitude spetrum is also called as Magnitude spetrum

Phase spetrum



fourier series is used to look at the frequency information of the periodic signal

<u>symmetric condition</u>:

① <u>Even symmetry</u>:-

If $x(t) = x(-t)$

Then $x(t)$ posses even symmetry

Graphically if it is symmetric above y-axis then signal posses even symmetric nature.

$$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\,dt \qquad \int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{even} \\ 0 & \text{odd} \end{cases}$$

$t_0 = -T/2$

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt$$

$$\boxed{a_0 = \frac{2}{T}\int_{0}^{T/2} x(t)\,dt}$$

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos(n\omega_0 t)\,dt$$

$t_0 = -T/2$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(n\omega_0 t)\,dt$$

$$\boxed{a_n = \frac{4}{T}\int_{0}^{T/2} x(t)\cos(n\omega_0 t)\,dt} \qquad n = 1, 2 \dots \infty$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin(n\omega_0 t)\,dt$$

$$\boxed{b_n = 0}$$

* For even symmetry signals only a_0, a_n TFS coefficients are their, and $b_n = 0$ because it is symmetry about y axis ∵

→ TFS of even symmetric periodic signal contains only dc terms and cosine terms, sine terms are absent

∴ TFS ⇒ $\boxed{x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t)}$

② odd symmetry

If $x(-t) = -x(t)$

Then $x(t)$ posses odd symmetry

→ Graphically it is symmetric about origine

→ If wave-form symmetric about origine then wave form posses odd symmetric

$$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\,dt$$

$a_0 = -T/2$

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} \underbrace{x(t)}_{0}\,dt$$

$$\boxed{a_0 = 0}$$

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} \underbrace{x(t)}_{\text{Odd}}\ \underbrace{\cos(n\omega_0 t)}_{\text{even}}\,dt$$

$t_0 = -T/2$

$\boxed{a_n = 0}$

$$b_n = \frac{2}{T}\int_{-T/2}^{t_0+T/2} x(t)\sin(n\omega_0 t)\,dt$$

$$\boxed{b_n = \frac{4}{T}\int_0^{T/2} x(t)\sin(n\omega_0 t)\,dt}$$

The TFS of odd symmetric periodic signal contains only sine terms

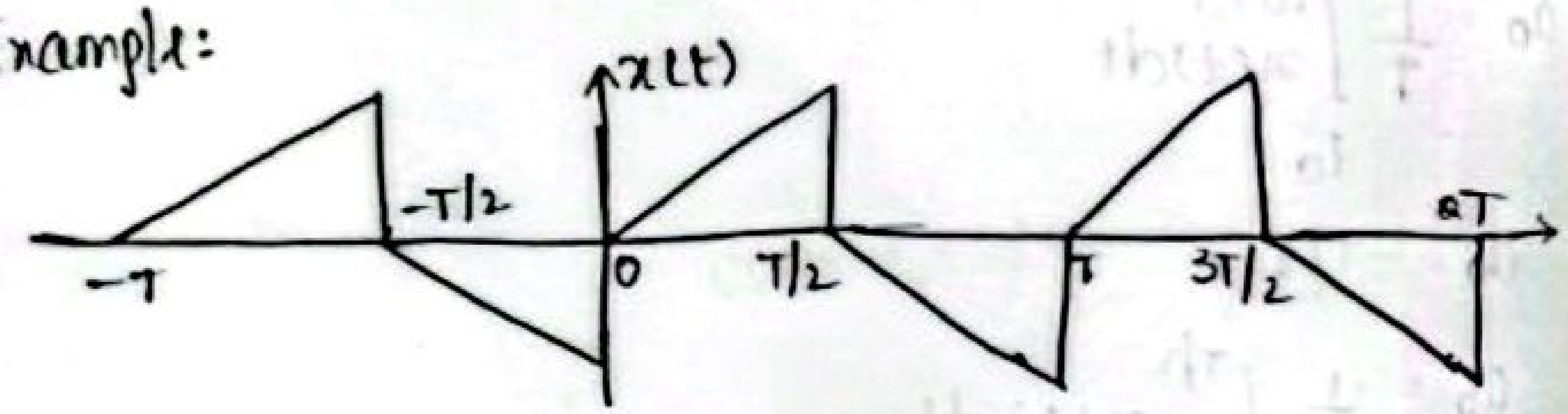
$\rightarrow$ TFS $\Rightarrow$ $\boxed{x(t) = \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)\,dt}$

alf-wavesymmtry

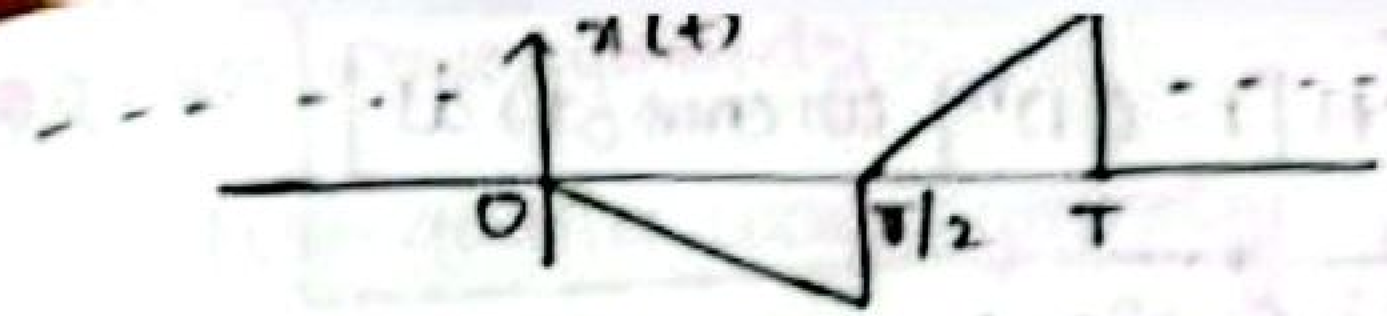
If $x(t) = -x(t \pm T/2)$

Then $x(t)$ posses Half-wave symmetry propertie

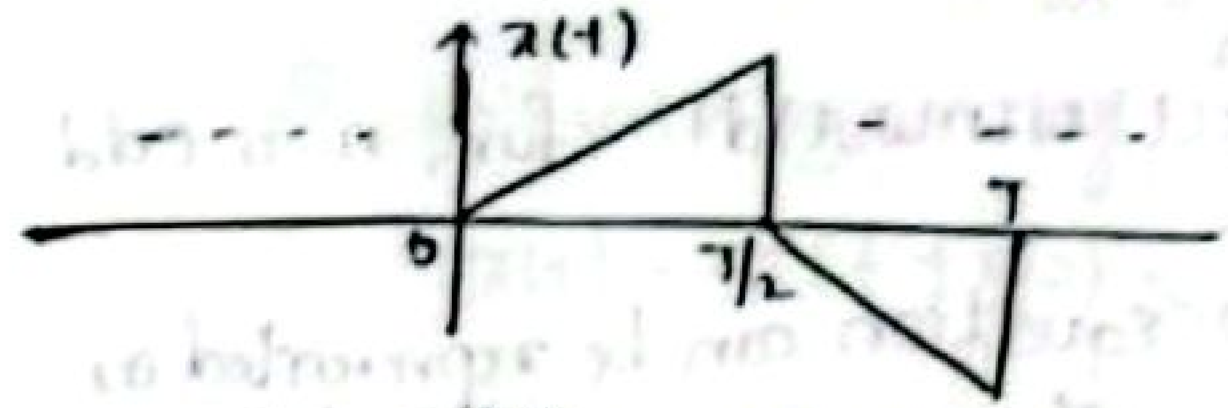
Example:



$x(t - T/2)$ (right shifting)



$-x(t-T/2)$ (Amplitude scaling)



$\therefore\ x(t) = -x(t - T/2)$

$$\Rightarrow a_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos(n\omega_0 t)\,dt$$

let $t_0 = 0$

$$a_n = \frac{2}{T}\int_0^T x(t)\cos(n\omega_0 t)\,dt$$

$$a_n = \frac{2}{T}\left[\int_0^{T/2} x(t)\cos(n\omega_0 t)\,dt + \int_{T/2}^{T} x(t)\cos(n\omega_0 t)\,dt\right]$$

let ~~$x(t) = -x(t+T/2)$~~ $t = t_1 + T/2 \Rightarrow t_1 = t - T/2$

only in 2nd part

$$= \frac{2}{T}\left[\int_0^{T/2} x(t)\cos(n\omega_0 t)\,dt + \int_0^{T/2} x(t_1 + T/2)\cos(n\omega_0 (t_1 + \tfrac{T}{2}))\,d\right.$$

$$a_n = \frac{2}{T}\left[\int_0^{T/2} x(t)\cos(n\omega_0 t)\,dt + \int_0^{T/2} x(t_1+T/2)\cos\left[n\omega_0 t_1 + \frac{n\omega_0 T}{2}\right]\right.$$

$$x(t) = -x(t + T/2)$$

$$a_n = \frac{2}{T}\left[\int_0^{T/2} x(t)\cos(n\omega_0 t)\,dt + \int_0^{T/2} -x(t)(-1)^n \cos(n\omega_0 t)\,dt\right]$$

$$\boxed{a_n = \frac{2}{T}\left[\int_0^{T/2} x(t)\left[1-(-1)^n\right]\cos(n\omega_0 t)\,dt\right]}$$

$$a_n = \begin{cases} 0 & \text{for, } n \text{ is even} \\ \frac{4}{T}\int_0^{T/2} x(t)\cos(n\omega_0 t)\,dt & \text{for, } n \text{ is odd} \end{cases}$$

$\Rightarrow$ The above two Equation can be represented as

$$a_{2n-1} = \frac{4}{T}\int_0^{T/2} x(t)\cos[(2n-1)\omega_0 t]\,dt$$

$\forall\, n = 1,2,3,4,5 \ldots$

Similarly

$$b_n = \frac{2}{T}\int_0^{T/2} x(t)\left[1-(-1)^n\right]\sin(n\omega_0 t)\,dt$$

$\forall\, n = 1,2,3,4,5 \ldots$

$$b_n = \begin{cases} 0 & \text{for, } n \text{ is even} \\ \frac{4}{T}\int_0^{T/2} x(t)\sin(n\omega_0 t)\,dt & \text{for, } n \text{ is odd} \end{cases}$$

The above two Equation can be represented as

$$b_{2n-1} = \frac{4}{T}\int_0^{T/2} x(t)\sin[(2n-1)\omega_0 t]\,dt$$

for $n = 1,2,3 \ldots$

TFS

$$x(t) = \sum_{n=1}^{\infty} a_{2n-1}\cos[(2n-1)\omega_0 t] + b_{2n-1}\sin[(2n-1)\omega_0 t]$$

for Half wave symmetry it contains both cosine and sine terms but they are odd harmonic for TFS.

Quarter wave symmetry:

→ Even quarter wave symmetry

① $x(t) = x(-t) \begin{cases} a_0, a_n \neq 0 \\ b_n = 0 \end{cases}$

② Half wave symmetry

$x(t) = -x(t \pm T/2) \begin{cases} a_{2n-1} \neq 0,\ a_0 = 0 \\ b_{2n-1} \neq 0 \end{cases}$

for even quarter wave symmetry it contain both even symmetry and Half wave symmetry

$x(t) = x(-t) = -x(t \pm T/2) \rightarrow a_{2n-1}$ coefficient present

∵ Even quarter wave periodic symmetry, whose TFS contains only odd harmonic of cosine terms

TFS

$$x(t) = \sum_{n=1}^{\infty} a_{2n-1}\cos((2n-1)\omega_0 t)$$

if $f(x) = f(2a-x)$ Then

$$\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$$

Due to Half wave symmetry

$$a_{2n-1} = \frac{4}{T}\int_0^{T/2} x(t)\cos[(2n-1)\omega_0 t]\,dt$$

$\forall\, n = 1,2,3,4 \ldots$

let $f(x) = x(t)\cos[(2n-1)\omega_0 t]$ — ①

$$f\left(\tfrac{T}{2}-t\right) = x\left(\tfrac{T}{2}-t\right)\cos\left[(2n-1)\omega_0\left(\tfrac{T}{2}-t\right)\right]$$

$$= x\left(\tfrac{T}{2}-t\right)\cos[(2n-1)\omega_0 t - (2n-1)\omega_0 t]$$

even $x(t) = x(-t)$

$$= x\left(t-\tfrac{T}{2}\right)\cos[\pi-(2n-1)\omega_0 t]$$

$$f\left(\tfrac{T}{2}-t\right) = -x\left(t-\tfrac{T}{2}\right)\cos[(2n-1)\omega_0 t]$$

$f(\frac{T}{2}-t) = x(t)\cos[(2n-1)\omega_0 ...]$

$\therefore f[\frac{T}{2}-t] = f(t)$

$\therefore$ from eqn ①

$$\boxed{a_{2n-1} = \frac{8}{T}\int_0^{T/4} x(t)\cos[(2n-1)\omega_0 t]\,dt}$$

→ odd quarter wave symmetry

① $x(t) = -x(-t)$ < $a_0, a_n = 0$; $b_n \neq 0$

② Halfwave symmetry $x(t) = -x(t \mp T/2)$ < $b_{2n-1}, a_{2n-1} \neq 0$; $a_0 = 0$

Odd quarter wave symmetry possess odd symmetry and Halfwave symmetry

$\therefore x(t) = -x(-t) = -x(t \pm T/2) \rightarrow b_{2n-1}$ coefficient present

odd quarter wave symmetry where TFS contain only odd harmonic of sine terms

TFS

$$x(t) = \sum_{n=1}^{\infty} b_{2n-1}\sin((2n-1)\omega_0 t)$$

due to half wave symmetry

$$b_{2n-1} = \frac{4}{T}\int_0^{T/2} x(t)\sin[(2n-1)\omega_0 t]\,dt$$

$f(t) = x(t)\sin[(2n-1)\omega_0 t]$

$f[\frac{T}{2}-t] = x(\frac{T}{2}-t)\sin[(2n-1)\omega_0(\frac{T}{2}-t)]$

$= x(\frac{T}{2}-t)\sin[(2n-1)\omega_0\frac{T}{2} - ...]$

$= -x(t-\frac{T}{2})\sin[\pi-(2n-1)\omega_0 t]$

$f(\frac{T}{2}-t) = +x(t-\frac{T}{2})\sin[(2n-1)\omega_0 t]$

$\therefore f(\frac{T}{2}-t) = f(t)$

$$\boxed{\therefore b_{2n-1} = \frac{8}{T}\int_0^{T/4} x(t)\sin[(2n-1)\omega_0 t]\,dt}$$

Parsevals Theorem

Parsevals Theorem use to compute the average power of the periodic signal

→ All periodic signals are power signals

power signal < Periodic signal ; Random signal

$$P_{av} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|x(t)|^2\,dt$$

Average power of the periodic signal

$$\star\ P_{av} = \frac{1}{T}\int_{-T/2}^{T/2}|x(t)|^2\,dt = \frac{1}{T}\int_0^T |x(t)|^2\,dt$$

where T is fundamental period

TFS

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n\cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n\sin(n\omega_0 t)$$

$$P_{av} = \frac{1}{T}\int_{t_0}^{t_0+T} x^2(t)\,dt$$

$$= \frac{1}{T}\int_{t_0}^{t_0+T}\left[a_0 + \sum_{n=1}^{\infty}\cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n\sin(n\omega_0 t)\right]dt$$

$$\Rightarrow \frac{1}{T}\int_{t_0}^{t_0+T}\Big[a_0^2+\Big(\sum_{n=1}^{\infty}a_n\cos(n\omega_0 t)\Big)^2+\Big(\sum_{n=1}^{\infty}b_n\sin(n\omega_0 t)\Big)^2+$$

$$2a_0\sum_{n=1}^{\infty}a_n\cos(n\omega_0 t)+2\sum_{n=1}^{\infty}a_n b_n\cos(n\omega_0 t)\sin(n\omega_0 t)$$

$$+\,2\sum_{n=1}^{\infty}b_n a_0\sin(n\omega_0 t)\Big]\,dt$$

As we know that sine and cosine are orthogonal signals

$\because$ orthogonal signal

① $$\int_{t_0}^{t_0+T}\cos(n\omega_0 t)\cos(m\omega_0 t)\,dt=\begin{cases}0 & \text{for } n\neq m\\ T/2 & \text{for } n=m\end{cases}$$

② $$\int_{t_0}^{t_0+T}\sin(n\omega_0 t)\sin(m\omega_0 t)\,dt=\begin{cases}0 & \text{for } n\neq m\\ T/2 & \text{for } n=m\end{cases}$$

$$\int_{t_0}^{t_0+T}\sin(n\omega_0 t)\sin(m\omega_0 t)\,dt=0$$

③ $$\int_{t_0}^{t_0+T}\cos(n\omega_0 t)\sin(m\omega_0 t)\,dt=0$$

$$P_{av}=\frac{1}{T}\int_{t_0}^{t_0+T}a_0^2\,dt+\frac{1}{T}\int_{t_0}^{t_0+T}\sum_{n=1}^{\infty}(a_n\cos(n\omega_0 t))^2dt+\frac{1}{T}\int_{t_0}^{t_0+T}\sum_{n=1}^{\infty}(b_n\sin(n\omega_0 t))^2dt$$

$$P_{av}=\frac{1}{T}\int_{t_0}^{t_0+T}a_0^2\,dt+\frac{1}{T}\int_{t_0}^{t_0+T}\sum_{n=1}^{\infty}a_n^2\cos^2(n\omega_0 t)\,dt+\frac{1}{T}\int_{t_0}^{t_0+T}\sum_{n=1}^{\infty}b_n^2\sin^2(n\omega_0 t)$$

$$P_{av}=a_0^2+\frac{1}{T}\sum_{n=1}^{\infty}a_n^2\frac{T}{2}+\frac{1}{T}\sum_{n=1}^{\infty}b_n^2\frac{T}{2}$$

$$\boxed{P_{av}=a_0^2+\frac{1}{2}\sum_{n=1}^{\infty}a_n^2+b_n^2}$$

Average power to the periodic signal

$$P_{av}=\frac{1}{T}\int_{t_0}^{t_0+T}x(t)^2\,dt=a_0^2+\frac{1}{2}\sum_{n=1}^{\infty}a_n^2+b_n^2=\sum_{n=-\infty}^{\infty}|C_n|^2$$

Ex:- $x(t)=3+4\cos(3t)$ find P_{av}

$$\omega_0=3$$

$$T=\frac{2\pi}{\omega_0}=\frac{2\pi}{3}$$

$$x(t)=a_0+\sum_{n=1}^{\infty}a_1\cos(\omega_0 t)+b_1\sin(\omega_0 t)$$

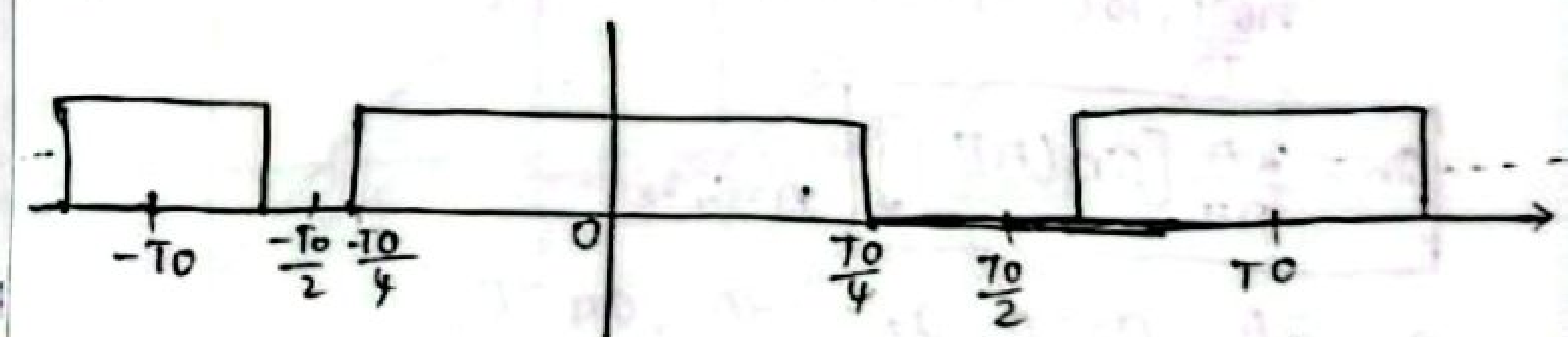
$$=a_0+\sum_{n=1}^{\infty}a_n\cos(3t)+b_1\sin(3t)$$

$$a_0=3,\ a_1=4$$

$$P_{av}=a_0^2+\frac{1}{2}a_1^2\Rightarrow 9+\frac{16}{2}=17$$

$$\text{rms}=\sqrt{P_{av}}=\sqrt{17}$$

Q Find FFS for following periodic signals.



The wave posses even symmetry (The wave is symmetric about y-axis). only a_0 and a_n coefficient are present

$b_n=0$

$$a_0=\frac{1}{T}\int_{t_0}^{t_0+T}x(t)\,dt \quad (\text{dc (or) average value})$$

Because of Even symmetry

$a_0 = \frac{2}{T}\int_0^{T/2} x(t)\,dt = \frac{2}{T_0}\left[\int_0^{T_0/4} A\,dt + \int_{T_0/4}^{T_0/2} 0\,dt\right]$

$= \frac{2}{\cancel{T_0}}\left[\frac{A\cancel{T_0}}{4}\right]$

$\boxed{a_0 = \frac{A}{2}}$

$a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos(n\omega_0 t)\,dt$

$a_n = \frac{4}{T_0}\left[\int_0^{T_0/4} A\cos(n\omega_0 t)\,dt + \int_{T_0/4}^{T_0/2} 0\,dt\right]$

$= \frac{4A}{T_0}\left[\frac{\sin(n\omega_0 t)}{n\omega_0}\right]_0^{T_0/4}$

$= \frac{4A}{T_0}\left[\frac{\sin(n\omega_0 \frac{T_0}{4})}{n\omega_0} - 0\right]$

$= \frac{4A}{n\omega_0 T_0}\left[\sin\left(n\frac{2\pi}{\cancel{T_0}}\times\frac{\cancel{T_0}}{4}\right)\right]$

$= \frac{4A}{n\frac{2\pi}{T_0}\times T_0}\left[\sin\left(\frac{n\pi}{2}\right)\right]$

$\boxed{a_n = \frac{2A}{n\pi}\left[\sin\left(\frac{n\pi}{2}\right)\right]}$ $n = 1, 2, \cdots$

$a_1 = \frac{2A}{\pi}$, $a_2 = 0$, $a_3 = \frac{-2A}{3\pi}$, $a_4 = 0$,

$a_5 = \frac{2A}{5\pi}$ ....

TFS

$x(t) = a_0 + \sum_{n=1}^{\infty} a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t)$

$x(t) = \frac{A}{2} + a_1\cos(\omega_0 t) + a_2\cos(2\omega_0 t) + a_3\cos(3\omega_0 t) + \cdots$

$x(t) = \frac{A}{2} + \frac{2A}{\pi}\left[\cos(\omega_0 t) - \frac{1}{3}\cos(3\omega_0 t) + \frac{1}{5}\cos(5\omega_0 t) - \frac{1}{7}\cos(7\omega_0 t) \cdots\right]$

$P_{av} = \frac{1}{T}\int_{t_0}^{t_0+T} x^2(t)\,dt$

$= \frac{1}{T}\int_{-T/2}^{T/2} x^2(t)\,dt$ Becaus of Even symmetry

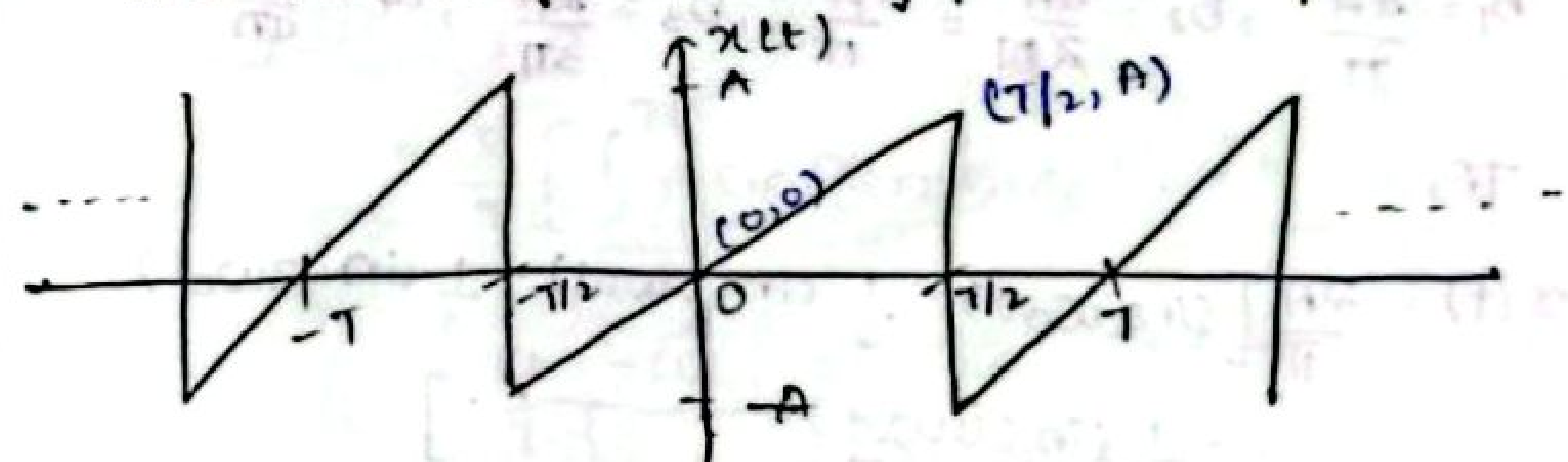
$= \frac{2}{T}\int_0^{T/2} x^2(t)\,dt$

$= \frac{2}{T_0}\left[\int_0^{T_0/4} A^2\,dt \int_{T_0/4}^{T_0/2} 0\,dt\right]$

$= \frac{2}{\cancel{T_0}} \times A^2\left(\frac{\cancel{T_0}}{4}\right)$

$\boxed{P_{av} = \frac{A^2}{2}}$

1. Find the TFs for following periodic signal.



Odd symmetry $a_0 = 0$, $a_n = 0$, $b_n \neq 0$

$b_n = \frac{4}{T}\int_0^{T/2} x(t)\sin(n\omega_0 t)\,dt$

$b_n = \frac{4}{T_0}\left[\int_0^{T_0/4}\right.$ $y = mx + c$

$m = \frac{A - 0}{\frac{T}{2} - 0} = \frac{2A}{T}$

$x(t) = \frac{2A}{T} t$

$$b_n = \frac{4}{T}\left(\frac{2A}{T}\right)\left[\int_0^{T/2} t \sin(n\omega_0 t)\, dt\right]$$

$$= \frac{8A}{T^2}\left[\left[\frac{-t\cos(n\omega_0 t)}{n\omega_0}\right]_0^{T/2} - \int_0^{T/2} \frac{-\cos(n\omega_0 t)}{n\omega_0} dt\right]$$

$$= \frac{8A}{T^2}\left[\left[\frac{-\frac{T}{2}\cos\left(\frac{n\omega_0 T}{2}\right)}{n\omega_0} + 0\right] + \left[\frac{\sin(n\omega_0 t)}{(n\omega_0)^2}\right]_0^{T/2}\right]$$

$$= \frac{-8A}{T^2}\left[\frac{\frac{T}{2}\cos\left(n\frac{2\pi}{T}\times\frac{T}{2}\right)}{n\omega_0}\right]$$

$$= -\frac{8A}{T(2)} \frac{\cos(n\pi)}{n\frac{(2\pi)}{T}}$$

$$\boxed{b_n = -\frac{2A}{n\pi}\cos(n\pi)} \quad n = 1, 2, 3 \ldots\ldots$$

$b_1 = \frac{2A}{\pi}$, $b_2 = \frac{-2A}{2\pi} = -\frac{A}{\pi}$, $b_3 = \frac{2A}{3\pi}$, $b_4 = -\frac{2A}{4\pi}$

TFS:-

$$x(t) = \frac{2A}{\pi}\left[\sin(\omega_0 t) - \frac{1}{2}\sin(2\omega_0 t) + \frac{1}{3}\sin(3\omega_0 t) - \frac{1}{4}\sin(4\omega_0 t) + \ldots\ldots\right]$$

$$P_{avg} = \frac{1}{T}\int_{t_0}^{t_0+T} x^2(t)\, dt$$

$$= \frac{2}{T}\int_0^{T/2} x^2(t)\, dt$$

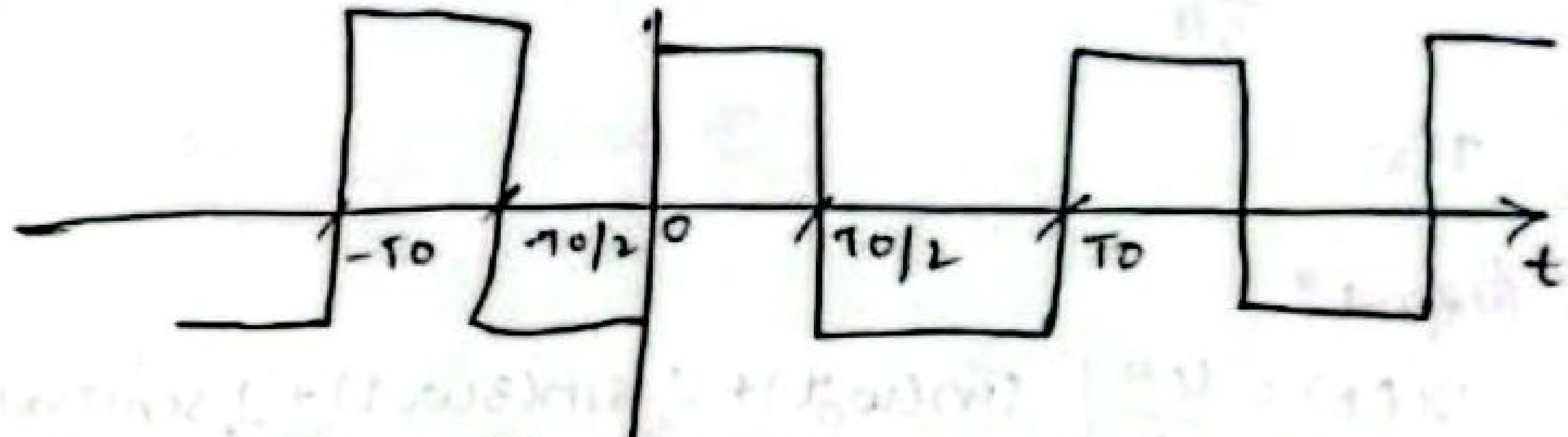
$$= \frac{2}{T}\int_0^{T/2} \left(\frac{2A}{T} t\right)^2 dt$$

$$= \frac{4A^2}{T^3}\left(\frac{t^3}{3}\right)_0^{T/2}$$

$$= \frac{4A^2}{T^3}\left(\frac{T^3}{8}\right)$$

$$\boxed{P_{av} = \frac{A^2}{2}}$$

Q) Find the TFS for following periodic signal



This possess odd quarter wave symmetry whose TFS contains odd harmonic of sine terms

i.e., $x(t) = \sum_{n=1}^{\infty} b_{2n-1} \sin[(2n-1)\omega_0 t]$

$$b_{2n-1} = \frac{8}{T}\int_0^{T/4} x(t)\sin[(2n-1)\omega_0 t]\, dt$$

$$= \frac{8}{T}\left[\int_0^{T/4} A\sin[(2n-1)\omega_0 t)\, dt\right.$$

$$= \frac{8A}{T}\left[\frac{-\cos((2n-1)\omega_0 t)}{(2n-1)\omega_0}\right]_0^{T/4}$$

$$= \frac{-8A}{T}\left[\frac{\cos(2n-1)\frac{2\pi}{T}\left(\frac{T}{4}\right)}{(2n-1)\frac{2\pi}{T}}\right]$$

$$= \frac{-8A}{T}\left[\frac{\cos[(2n-1)\frac{\pi}{2}] - \cos 0}{(2n-1)\frac{2\pi}{T}}\right]$$

$$b_{2n-1} = \frac{4A}{(2n-1)\pi}\left[1 - \cos(2n-1)\frac{\pi}{2}\right]$$

$n=1$: $b_1 = \frac{4A}{\pi}[1-0] = \frac{4A}{\pi}$

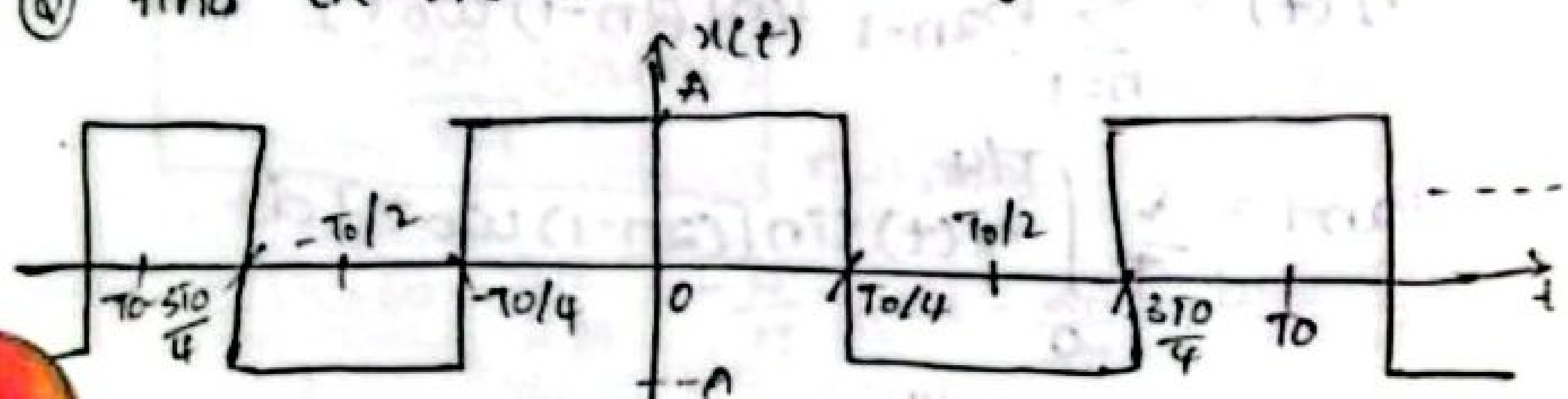
$n=2$: $b_3 = \frac{4A}{3\pi}[1-0] = \frac{4A}{3\pi}$

$b_5 = \frac{4A}{5\pi}$

TFS:-

$$x(t) = \frac{4A}{\pi}\left[\sin(\omega_0 t) + \frac{1}{3}\sin(3\omega_0 t) + \frac{1}{5}\sin(5\omega_0 t) + \cdots\cdots\right]$$

Q find the TFS for the following periodic signal



This posses even symmetry and Half wave symmetry

∴ This contains odd harmonic of cosine terms

$$\because \text{TFS} :\to a_{2n-1} = \frac{8}{T}\int_0^{T/4} x(t)\cos((2n-1)\omega_0 t)\,dt$$

$$= \frac{8}{T_0}\int_0^{T_0/4} A\cos(2n-1)\omega_0 t\,dt$$

$$= \frac{8A}{T_0}\left[\frac{\sin(2n-1)\omega_0 t}{(2n-1)\omega_0}\right]_0^{T_0/4}$$

$$\Rightarrow \frac{8A}{T_0}\left[\frac{\sin(2n-1)\omega_0\frac{T_0}{4}}{(2n-1)\omega_0}\right]$$

$$\Rightarrow \frac{8A}{T_0}\;\frac{\sin(2n-1)\frac{2\pi}{T_0}\cdot\frac{T_0}{4}}{(2n-1)\frac{2\pi}{T_0}}$$

$$= \frac{8A}{2\pi}\;\frac{\sin\left((2n-1)\frac{\pi}{2}\right)}{(2n-1)}$$

$$= \frac{4A}{\pi(2n-1)}\sin\left((2n-1)\frac{\pi}{2}\right) \qquad n=1,2,\ldots\ldots$$

for $n\geq 1$

$a_1 = \frac{4A}{\pi}$

$a_3 = -\frac{4A}{3\pi}$

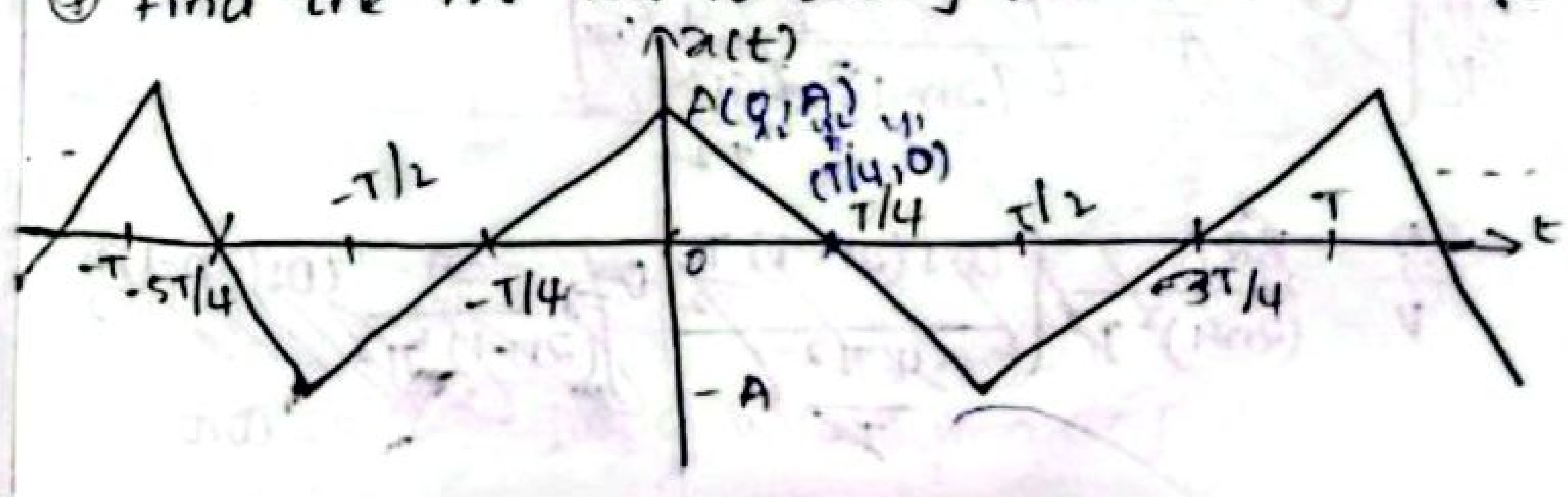
$a_5 = \frac{4A}{5\pi}$

$a_7 = -\frac{4A}{7\pi}$ ......

TFS :-

$$x(t) = \frac{4A}{\pi}\left[\cos(\omega_0 t) - \frac{1}{3}\cos(3\omega_0 t) + \frac{1}{5}\cos(5\omega_0 t) - \frac{1}{7}\cos(7\omega_0 t)\cdots\cdots\right]$$

Q find the TFS for following periodic signal :



This signal posses Even symmetry and half wave symmetry ∵ It is Even quarter wave symmetry

$$\therefore \text{TFS} :- x(t) = \sum_{n=1}^{\infty} a_{2n-1} \cos((2n-1)\omega_0 t)$$

$$a_{2n-1} = \frac{8}{T}\int_0^{T/4} x(t) \cos((2n-1)\omega_0 t)\,dt$$

$$y = mx + c$$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{A - 0}{0 - \frac{T}{4}} = \frac{-4A}{T}$$

$$\therefore x(t) = -\frac{4A}{T}t + A$$

$$= A\left(-\frac{4t}{T} + 1\right)$$

$$a_{2n-1} = \frac{8A}{T}\int_0^{T/4}\left(-\frac{4t}{T} + 1\right)\cos((2n-1)\omega_0 t)\,dt$$

$$= \frac{8A}{T}\left[\left[\left(-\frac{4t}{T}+1\right)\frac{\sin((2n-1)\omega_0 t)}{(2n-1)\omega_0}\right]_0^{T/4} - \int_0^{T/4} -\frac{4}{T}\,\frac{\sin(2n-1)\omega_0 t}{(2n-1)\omega_0}\,dt\right]$$

$$= \frac{8A}{T}\left[\left[\frac{-4\left(\frac{T}{4}\right)}{T}+1\ \frac{\sin(2n-1)\omega_0 \frac{T}{4}}{2n-1\omega_0}\right] + \frac{4}{T}\left[\frac{\cos(2n-1)\omega_0 t}{((2n-1)\omega_0)^2}\right]_0^{T/4}\right]$$

$$\Downarrow 0$$

$$\cancel{= \frac{8A}{T}\left[+0 - \frac{4}{T}\,\frac{\cos(2n-1)\frac{2\pi}{T}\frac{T}{4}-\cos 0}{(2n-1)^2\frac{4\pi^2}{T^2}}\right]}$$

$$\cancel{= \frac{8A}{T} - \frac{8A}{(2n-1)^2\pi^2}\left\{\frac{\cos(2n-1)\frac{\pi}{2} - \cos 0}{\frac{4\pi^2}{T^2}}\right\} \frac{-8A}{(2n-1)^2\pi^2}\cos(2n-1)\frac{\pi}{2} - \cos 0}$$

$$= \frac{-8A}{(2n-1)\omega_0 T}\left[+\frac{4}{T}\,\frac{\cos(2n-1)\frac{\pi}{2}}{(2n-1)\omega_0} - \cos 0\right]$$

$$= \frac{-32A}{(2n-1)^2\omega_0^2 T^2}\left[\cos(2n-1)\frac{\pi}{2} - \cos 0\right]$$

$$= \frac{-32A}{(2n-1)^2\frac{4\pi^2}{T^2}(T^2)}\left[\cos(2n-1)\frac{\pi}{2} - \cos 0\right]$$

$$= \frac{-8A}{(2n-1)^2\pi^2}[0 - 1]$$

$$a_{2n-1} = \frac{8A}{(2n-1)^2\pi^2}$$

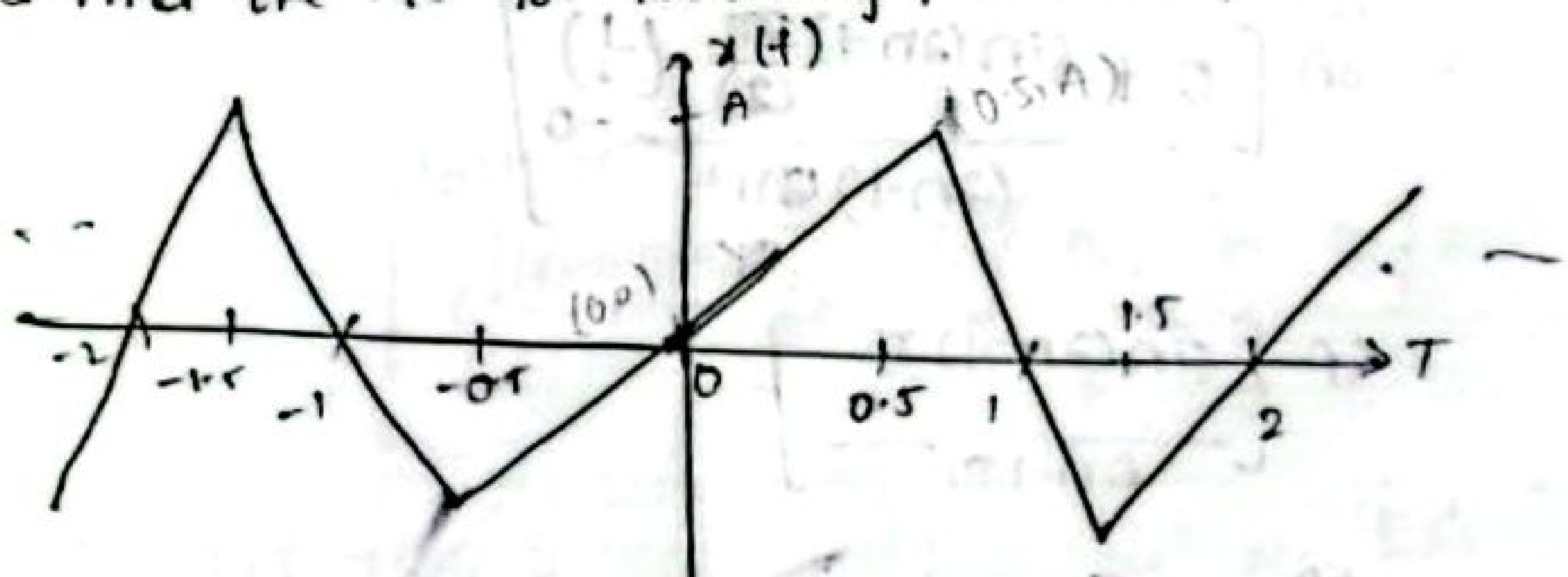
$$a_1 = \frac{8A}{\pi^2}$$

$$a_3 = \frac{8A}{9\pi^2}$$

$$a_5 = \frac{8A}{25\pi^2} \ldots$$

$$x(t) = \frac{8A}{\pi^2}\left[\cos(\omega_0 t) + \frac{1}{3^2}\cos(3\omega_0 t) + \frac{1}{5^2}\cos(5\omega_0 t) + \frac{1}{7^2}\cos(7\omega_0 t) + \cdots\right]$$

③ Find the TFS for following periodic signal



This is odd quarter wave symmetry

TFS : $x(t) = \sum_{n=1}^{\infty} b_{2n-1} \sin((2n-1)\omega_0 t)$

$$b_{2n-1} = \frac{8}{T}\int_0^{2} x(t)\sin((2n-1)\omega_0 t)$$

$$x_1 = mx + c$$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{A-0}{0.5-0} = \frac{A}{0.5}$$

$$x(t) = \frac{A}{0.5}t = 2At$$

$$b_{2n-1} = \frac{8}{2}\int_0^{0.5} 2At\,\sin((2n-1)\omega_0 t)\,dt$$

$$= \frac{16A}{2}\int_0^{0.5} t\,\sin((2n-1)\omega_0 t)\,dt$$

$$= \frac{16A}{2}\left[\left[t\frac{-\cos((2n-1)\omega_0 t}{(2n-1)\omega_0}\right]_0^{0.5} - \int_0^{0.5}\frac{-\cos(2n-1)\omega_0 t}{(2n-1)\omega_0}dt\right]$$

$$= \frac{8A}{8}\left[-\frac{0.5\cos(2n-1)\omega_0 t}{(2n-1)\omega_0} - 0\right] + \left[\frac{\sin(2n-1)\omega_0 t}{(2n-1)\omega_0^2}\right]_0^{0.5}$$

$$= 8A\left[0 + \frac{\sin(2n-1)\frac{2\pi}{(2)}\cdot\left(\frac{1}{2}\right)}{(2n-1)\frac{4\pi^2}{4}} - 0\right]$$

$$= 8A\left[\frac{\sin(2n-1)\frac{\pi}{2}}{(2n-1)\pi^2}\right]$$

$$= \frac{8A}{(2n-1)^2\pi^2}\left[\sin(2n-1)\frac{\pi}{2}\right]$$

$$\therefore b_{2n-1} = \frac{8A}{(2n-1)^2\pi^2}\left[\sin(2n-1)\frac{\pi}{2}\right]$$

$$b_1 = \frac{8A}{\pi^2}$$

$$b_3 = \frac{-8A}{3^2\pi^2}$$

$$b_5 = \frac{8A}{5^2\pi^2}$$

$$b_7 = \frac{-8A}{7^2\pi^2}$$

$$\therefore x(t) = \frac{8A}{\pi^2}\left[\sin(\pi t) - \frac{1}{3^2}\sin(3\pi t) + \frac{1}{5^2}\sin(5\pi t) - \frac{1}{7^2}\sin(7\pi t) + \cdots\right]$$

Exponential foreior series (EFS)

EFs, is complex function (combination of any two is orthogonal) 2

EFS, consist of $\{\cdots e^{-j3\omega_0 t}, e^{-j2\omega_0 t}, e^{-j\omega_0 t}, 1, e^{j\omega_0 t}, e^{j2\omega_0 t}, e^{j3\omega_0 t}\cdots\}$

$$\int_{t_0}^{t_0+T} e^{jn\omega_0 t}\left(e^{jm\omega_0 t}\right)^* dt = \int_{t_0}^{t_0+T} e^{jn\omega_0 t}\cdot e^{-jm\omega_0 t}\,dt$$

$$= \int_{t_0}^{t_0+T} e^{j(n-m)\omega_0 t}\,dt$$

$$\therefore \int_{t_0}^{t_0+T} e^{j(n-m)\omega_0 t}\,dt = \begin{cases} 0 ; & \text{for } n \neq m \\ T ; & \text{for } n = m \end{cases} \quad -(1)$$

(sine, cosine)

For TFS their is real functions. But for EFs complex-function.

$x(t) = \dots - + C_{-2}e^{-j2\omega_0 t} + C_{-1}e^{-j\omega_0 t} + C_0 + C_1 e^{j\omega_0 t} + C_2 e^{j2\omega_0 t} + \dots$

↓

Real periodic signal with period T

Forward basis function

$$\boxed{x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} C_n e^{jn\frac{2\pi}{T}t} \to \text{EFs}}$$

where C_n is called fourier series coefficient.

This equation is called as synthesis equation

perform integral inner product on both sides, where

$$\int_{t_0}^{t_0+T} x(t) e^{-jm\omega_0 t} dt$$ -ve Because of complex

$$= \sum_{n=\infty}^{\infty} C_n \int_{t_0}^{t_0+T} e^{jn\omega_0 t} \cdot e^{-jm\omega_0 t} dt$$ m, n values ranges from $-\infty$ to ∞

from ① $\Rightarrow$ $$\boxed{C_m T = \int_{t_0}^{t_0+T} x(t) \cdot e^{-jm\omega_0 t} dt}$$

$$C_m = \frac{1}{T}\int_{t_0}^{t_0+T} x(t) \cdot e^{-jm\omega_0 t} dt$$

m is replaced with n

$$\boxed{\therefore C_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t) \cdot e^{-jm\omega_0 t} dt}$$ Inverse basis function

This "C_n" is called as fourier series coefficient of $x(t)$

And This eqn is called as Analysis Equation

C_n is always complex [In nature]

Since C_n is always complex

$\therefore$ It can be represented in polar form

$C_n = |C_n| e^{j\theta_n}$

$\theta_n = \angle C_n$

$$\boxed{C_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t) e^{jn\omega_0 t} dt}$$

C_{-n} is also a complex numbers

$\therefore$ It also can be represented in polar form

$C_{-n} = |C_{-n}| e^{j\theta_n}$

$\theta_{-n} = \angle C_{-n}$

$$C_n^* = \left[\frac{1}{T}\int_{t_0}^{t_0+T} x(t) e^{-jn\omega_0 t} dt\right]^*$$

Let (Example) $\left(\sum_{i=1}^{n} z_i\right)^* = \sum_{i=1}^{n} z_i^*$

$(z_1 + z_2 + z_3)^* = z_1^* + z_2^* + z_3^*$

$$C_n^* = \frac{1}{T}\int_{t_0}^{t_0+T} x^*(t) e^{+jn\omega_0 t} dt$$

Let $x(t)$ is a real signal

$x(t) = x^*(t)$

$$\boxed{C_n^* = \frac{1}{T}\int_{t_0}^{t_0+T} x(t) e^{jn\omega_0 t} dt = C_{-n}}$$

If $x(t)$ is real signal then $C_{-n} = C_n^*$

$\therefore$ C_n^* is also complex

$\therefore$ It can be represented in polar form

$C_n^* = [C_n^*]^* = |C_n^*| e^{-j\theta_n} = |C_n| e^{-j\theta_n}$

... signal

$C_{-n} = C_n^*$

$|C_{-n}|e^{j\theta_{-n}} = |C_n|e^{-j\theta_n}$

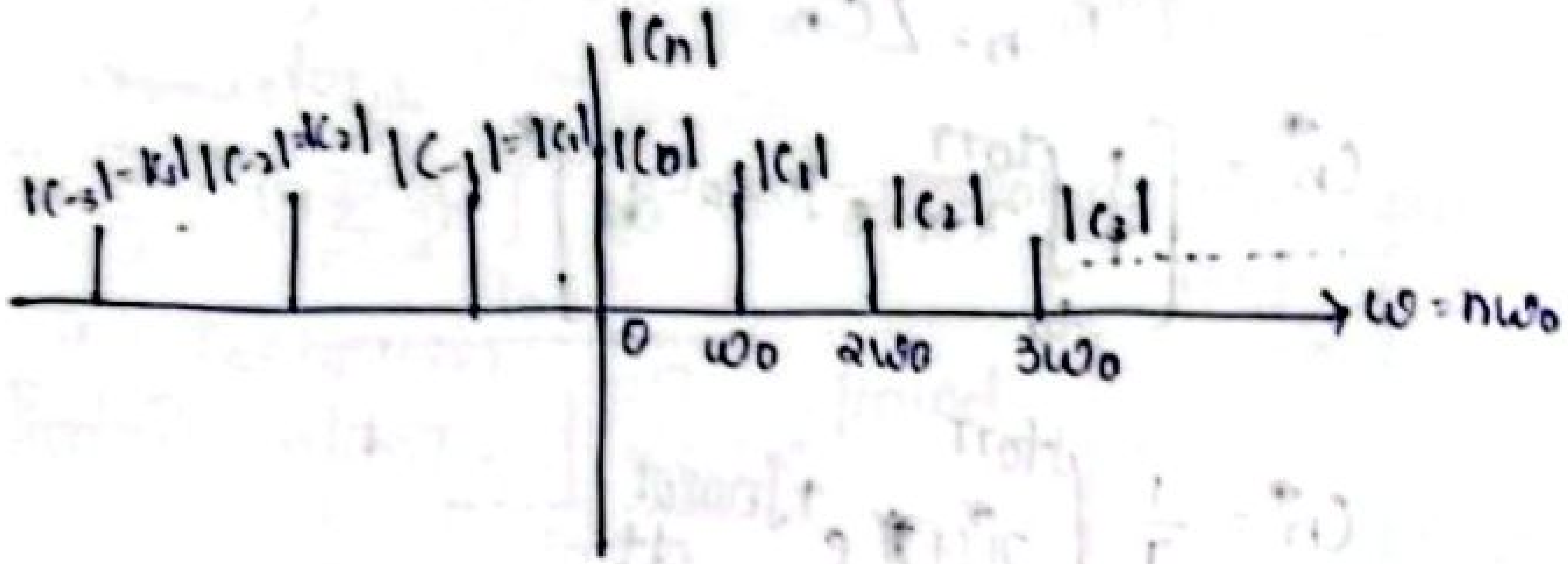
$\because \underbrace{|C_{-n}| = |C_n|}_{\text{even function}}$ and $\underbrace{\theta_{-n} = -\theta_n}_{\text{odd function}}$

Frequency spectrum:-

Frequency spectrum is combination of magnitude and phase spectrum

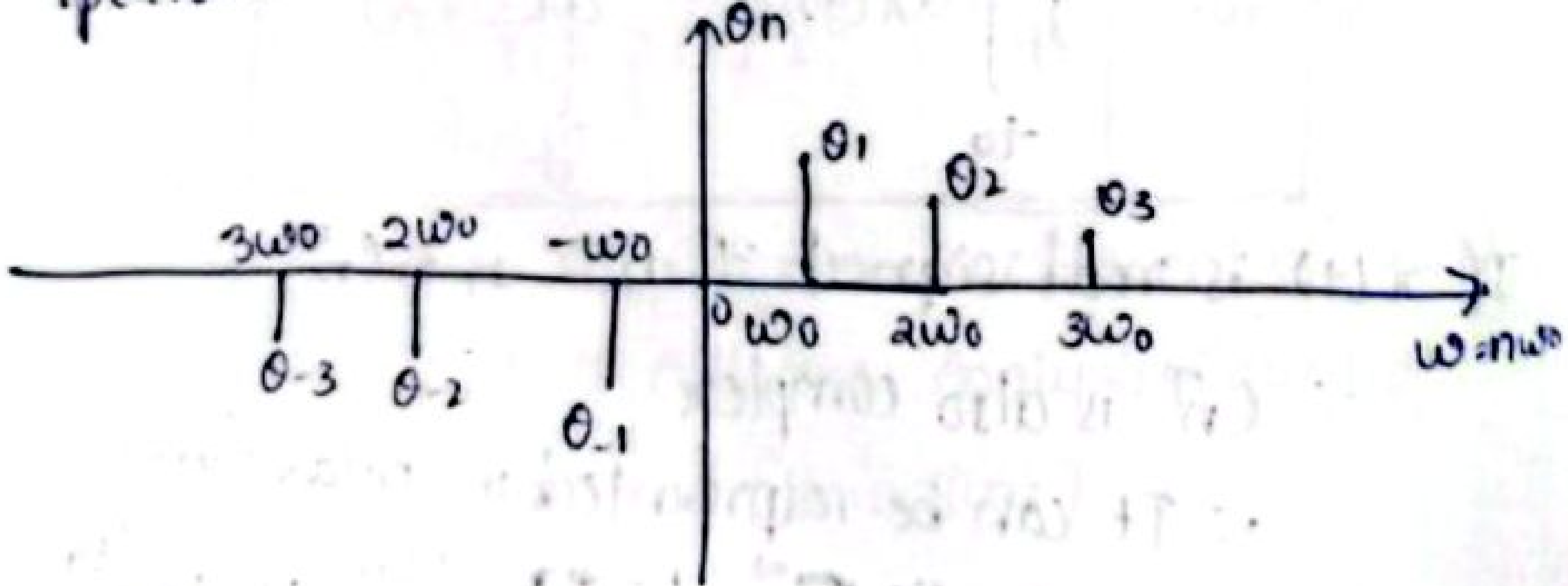
The plot b/w $|C_n|$ vs ω [Magnitude spectrum]

where $\omega = n\omega_0$



For $x(t)$ is real function magnitude spectrum is always even function of ω. It is discrete/line spectrum and two sided spectrum.

[Phase spectrum] The plot b/w θ_n vs ω is called phase spectrum



phase spectrum is ...

properties of :-

(i) Linearity property:-

If $x(t)$ is a periodic signal with fundamental period T whose fourier series is C_n

$$x(t) \xleftrightarrow{T} C_n$$

synthesis equation

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

Analysis equation

$$[x(t)] = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}\,dt$$

and $y(t) \xleftrightarrow{T} d_n$

$$\boxed{\therefore\ \alpha x(t) + \beta y(t) \xleftrightarrow{T} \alpha C_n + \beta d_n}$$

proof:-

Fourier series coefficient (FSC)

$$FSC[\alpha x(t) + \beta y(t)] = \frac{1}{T}\int_0^T [\alpha x(t) + \beta y(t)]e^{-jn\omega_0 t}\,dt$$

$$= \alpha \underbrace{\frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}\,dt}_{\text{Fourier series coefficient of } x(t) = C_n} + \beta \underbrace{\frac{1}{T}\int_0^T y(t)e^{-jn\omega_0 t}\,dt}_{FSC \text{ of } y(t) = d_n}$$

$\therefore FSC[\alpha x(t) + \beta y(t)] = \alpha C_n + \beta d_n$

$\therefore \alpha x(t) + \beta y(t) \longleftrightarrow \alpha C_n + \beta d_n$.

② Time scaling property

If $x(t) \xleftrightarrow[T]{} C_n$

then $x(\alpha t) \xleftrightarrow[T_1 = \frac{T}{|\alpha|}]{} C_n$

proof:

$$FSC[x(\alpha t)] = \frac{1}{T_1}\int_0^{T_1} x(\alpha t) e^{-jn\omega_1 t} dt_1 \qquad \omega_1 = \frac{2\pi}{T}$$

let $\alpha t = t_1 \Rightarrow dt = \frac{1}{\alpha} dt_1$

$t \to 0$, $t_1 \to 0$

$t \to T_1$, $t_1 \to T_1/\alpha$

$$= \frac{1}{T_1}\int_0^{\alpha T_1} x(t_1)\, e^{-jn\omega_0\left(\frac{t_1}{\alpha}\right)} \cdot \frac{1}{\alpha} dt_1$$

$$= \frac{1}{\alpha T_1}\int_0^{\alpha T_1} x(t_1) e^{-jn\omega_0 \frac{2\pi}{\alpha T_1} t_1} dt_1$$

$T_1 = T/\alpha$

$T = \alpha T_1$

$$= \frac{1}{T}\int_0^{T} x(t_1) e^{-jn\frac{2\pi}{(T)} t_1} dt_1$$

$\omega_0 \to$ fundamental frequency of $x(t)$

$t_1 \to t$

$$= \frac{1}{T}\int_0^{T} x(t) e^{-jn\omega_0 t} dt$$

$FSC[x(t)] = C_n$

③ Time reversal property

If $x(t) \xleftrightarrow[T]{} C_n$

then $x(-t) \xleftrightarrow[T]{} C_{-n}$

proof:

$$FSC[x(-t)] = \frac{1}{T}\int_0^T x(-t) e^{-jn\omega_0 t} dt$$

let $t_1 = -t \Rightarrow dt_1 = -dt$

$$= \frac{1}{T}\int_0^{-T} x(t_1) e^{jn\omega_0 t_1} (-dt_1)$$

$$= -\frac{1}{T}\int_0^{-T} x(t_1) e^{jn\omega_0 t} dt_1$$

$t_1 \to t$

$$= \frac{1}{T}\int_{-T}^{0} x(t) e^{-j(-n)\omega_0 t} dt$$

$x(t)$ is periodic

$[0, T]$ and $[-T, 0]$

$$= \frac{1}{T}\int_0^T x(t) e^{-j(-n)\omega_0 t} dt$$

$= C_{-n}$

④ Time shifting property:

If $x(t) \xleftrightarrow[T]{} C_n$

Then

i) $x(t - t_0) \xleftrightarrow[T]{} C_n e^{-jn\omega_0 t_0}$

proof:

$$FSC[x(t - t_0)] = \frac{1}{T}\int^T x(t - t_0) e^{-jn\omega_0 t}$$

let $t - t_0 = t_1$

$dt = dt_1$

$$= \frac{1}{T}\int_{-t_0}^{T-t_0} x(t_1) e^{-jn\omega_0(t_1+t_0)} dt_1$$

$$= \frac{1}{T}\int_{-t_0}^{T-t_0} x(t_1) e^{-jn\omega_0 t_1} \cdot e^{-jn\omega_0 t_0} dt_1$$

$$= e^{-jn\omega_0 t_0} \frac{1}{T}\int_{-t_0}^{T-t_0} x(t_1) e^{-jn\omega_0 t_1} dt_1$$

$t_1 \to t$

$$= e^{-jn\omega_0 t} \frac{1}{T}\int_{-t_0}^{T-t_0} x(t) e^{-jn\omega_0 t} dt$$

$$= e^{-jn\omega_0 t} \underbrace{\frac{1}{T}\int_0^T x(t) e^{-jn\omega_0 t} dt}$$

$FSC[x(t)] = C_n$

$\therefore FSC[x(t-t_0)] = C_n e^{-jn\omega_0 t}$

(ii) $x(t-t_0) \xleftrightarrow{T} C_n e^{-jn\omega_0 t_0}$

$x(t+t_0) \xleftrightarrow{T} C_n e^{jn\omega_0 t_0}$

⑤ frequency shifting property

If $x(t) \xleftrightarrow{T} C_n$

Then $x(t) e^{jm\omega_0 t} \xleftrightarrow{T} C_{n-m}$

proof:

$$FSC[x(t) e^{jm\omega_0 t}] = \frac{1}{T}\int_0^T x(t) e^{jm\omega_0 t} e^{-jn\omega_0 t} dt$$

$$= \frac{1}{T}\int_0^T x(t) e^{-j(n-m)\omega_0 t} dt$$

$$= C_{n-m}$$

⑥ Conjugation property

if $x(t) \xleftrightarrow{T} C_n$

Then

$x^*(t) \xleftrightarrow{T} C_{-n}^*$

proof:-

Fourier series coefficient of $x^*(t) = \frac{1}{T}\int_{-T/2}^{T/2} x^*(t) e^{-jn\omega_0 t} dt$

$$= \frac{1}{T}\int_{-T/2}^{T/2} (x(t) e^{jn\omega_0 t})^* dt$$

$$= \left[\frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-j(-n)\omega_0 t} dt\right]^*$$

$$\boxed{FSC[x^*(t)] = C_{-n}^*}$$

$x^*(t) \xleftrightarrow{T} C_{-n}^*$

⑦ ** Parsevals theorem:-

Pav of the periodic signal

$$P_{av} = \frac{1}{T}\int_{t_0}^{t_0+T} |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |C_n|^2 = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 + b_n^2$$

proof:-

$$P_{av} = \frac{1}{T}\int_{t_0}^{t_0+T} |x(t)|^2 dt = \frac{1}{T}\int_{t_0}^{t_0+T} x(t) x^*(t) dt$$

EFS

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$C_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t) e^{-jn\omega_0 t}\, dt$$

$$x^*(t) = \left(\sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}\right)^*$$

$$= \sum_{n=-\infty}^{\infty} \left(C_n e^{jn\omega_0 t}\right)^*$$

$$= \sum_{n=-\infty}^{\infty} C_n^* \left(e^{jn\omega_0 t}\right)^*$$

$$x^*(t) = \sum_{n=-\infty}^{\infty} C_n^* \left(e^{-jn\omega_0 t}\right)$$

$$\therefore P_{avg} = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\left[\sum_{n=-\infty}^{\infty} C_n^* e^{-jn\omega_0 t}\right] dt$$

$$= \sum_{n=-\infty}^{\infty} C_n^* \frac{1}{T}\int_{t_0}^{t_0+T} x(t) e^{-jn\omega_0 t}\, dt$$

$$= \sum_{n=-\infty}^{\infty} C_n^* . C_n$$

$$= \sum_{n=-\infty}^{\infty} |C_n|^2$$

$$\boxed{\therefore P_{avg} = \frac{1}{T}\int_{t_0}^{t_0+T} |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |C_n|^2}$$

* Q Find the complex exponential fourier series for the train of impluses. (or) Find FSC of $x(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

$$x(t) = \cdots + \underset{n=-2}{\delta(t+2T)} + \underset{n=-1}{\delta(t+T)} + \underset{n=0}{\delta(t)} + \underset{n=1}{\delta(t-T)} + \underset{n=2}{\delta(t-2T)} + \cdots$$

EFS:

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$C_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t) e^{-jn\omega_0 t}\, dt$$

let $t_0 = T/2$

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jn\omega_0 t}\, dt = \frac{1}{T} e^{-jn\omega_0 t}\Big|_{t=0}$$

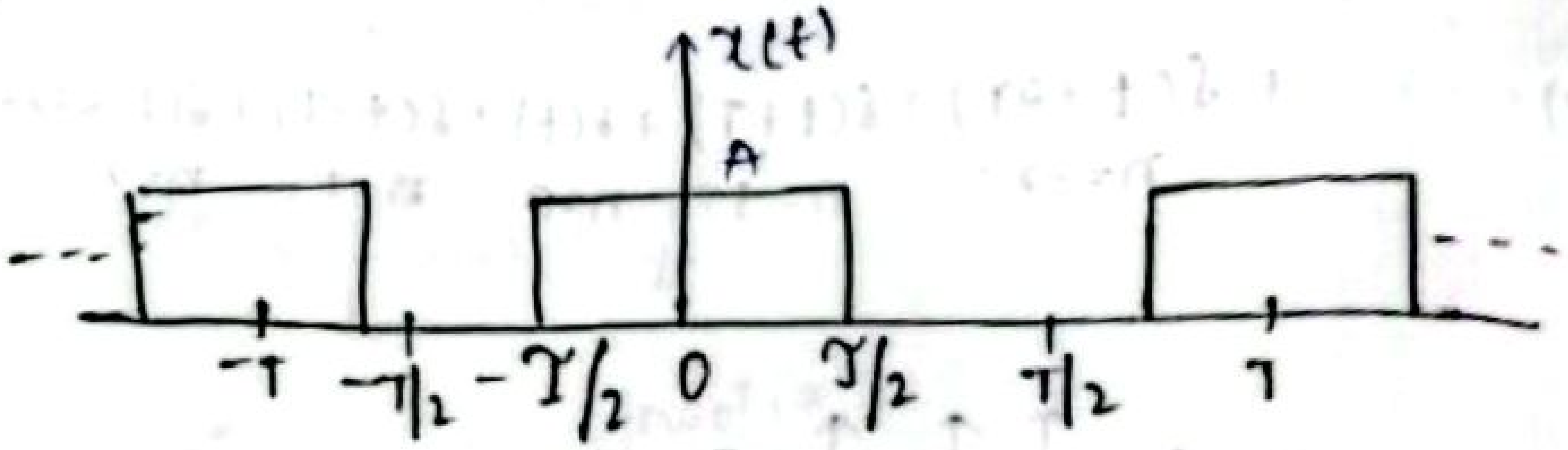
(Sampling property of an impluse $\int f(t)\delta(t)dt = f(t)|_{t=0}$)

$$C_n = \frac{1}{T} e^0 = \frac{1}{T}$$

$$\therefore \boxed{C_n = \frac{1}{T}}$$

EFS; $x(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}$

① find the EFS for the following periodic signal



EFS:-

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$C_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\, e^{-jn\omega_0 t}\, dt$$

let $t_0 = -T/2$

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{-\tau/2}^{\tau/2} A\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{A}{T}\left[\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\right]_{-\tau/2}^{\tau/2}$$

$$= -\frac{A}{Tjn\omega_0}\left[e^{-jn\omega_0 \frac{\tau}{2}} - e^{+jn\omega_0 \tau/2}\right]$$

$$= \frac{+A}{+jn\omega_0 T}\left[e^{jn\omega_0 \tau/2} - e^{-jn\omega_0 \tau/2}\right]$$

$$= \frac{A}{jn\omega_0 T}\left[2j\sin\left(\frac{n\omega_0 \tau}{2}\right)\right]$$

$$= \frac{2A\sin\left(\frac{n\omega_0 \tau}{2}\right)}{n\omega_0 T}$$

$Sa(\theta) = \frac{\sin\theta}{\theta}$ → Sampling

Multiply and divided with $\tau/2$

$$= 2A \cdot \frac{\sin\left(\frac{n\omega_0 \tau}{2}\right)}{n\omega_0 T \times \frac{\tau}{2}} \times \frac{\tau}{2}$$

$$= \frac{A\tau}{T}\,\frac{\sin\left(\frac{n\omega_0 \tau}{2}\right)}{\left(\frac{n\omega_0 \tau}{2}\right)}$$

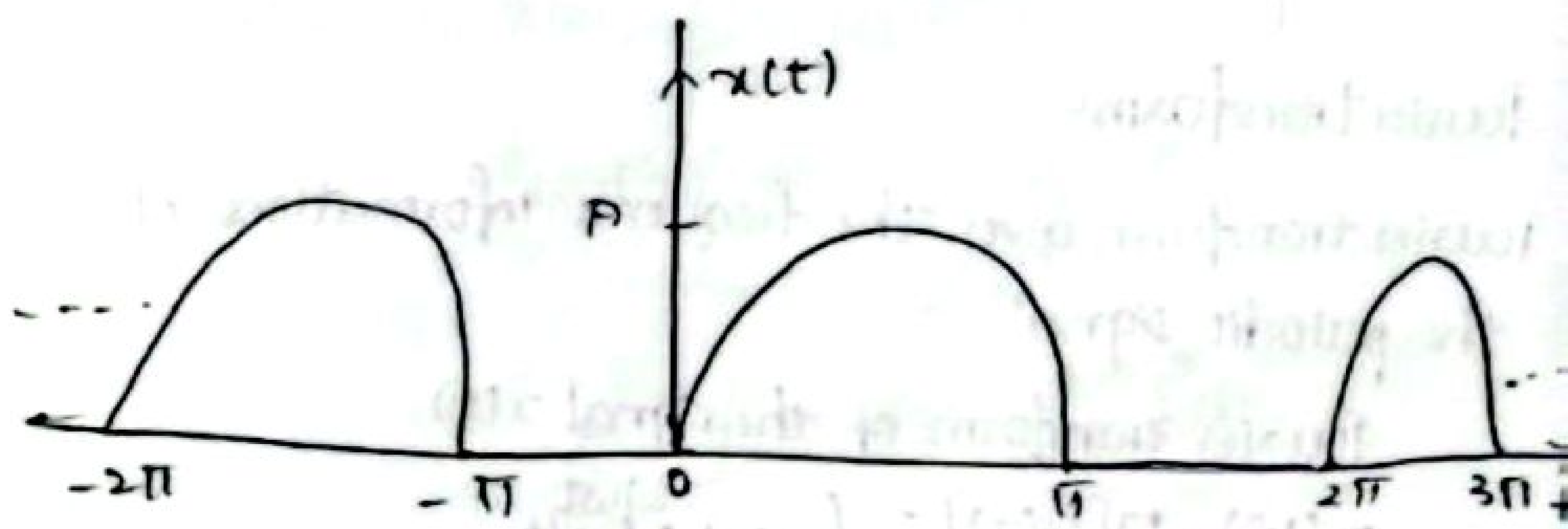
$$\boxed{C_n = \frac{A\tau}{T}\, Sa\left(\frac{n\omega_0 \tau}{2}\right)}$$

$\cancel{x(t)/=/T}$

$$\boxed{x(t) = \frac{A\tau}{T}\sum_{n=-\infty}^{\infty} Sa\left(\frac{n\omega_0 T}{2}\right) e^{jn\omega_0 t}}$$

② Find the EFS for half wave rectified sinusoidal

$$x(t) = \begin{cases} A\sin(t) & 0 \le t \le \pi \\ 0 & \pi \le t \le 2\pi \end{cases}$$



EFS:-

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$C_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\, e^{-jn\omega_0 t}\, dt$$

$t = 0$, $T = 2\pi$, $\omega_0 = 1$

$$C_n = \frac{1}{2\pi}\int_0^{2\pi} x(t)\, e^{-jnt}\, dt$$

$$= \frac{1}{2\pi}\int_0^{\pi} A\sin(t)\, e^{-jnt}\, dt$$

$$C_n = \frac{A}{2\pi}\int_0^{\pi} e^{-jnt}\sin(t)\, dt$$

$$\boxed{\int e^{ax}\sin(bx)\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin(bx) - b\cos(bx)\right]}$$

$a = -jn$, $b = 1$

$$C_n = \frac{A}{2\pi}\left(\frac{1+e^{-jn\pi}}{1-n^2}\right)$$

$$C_n = \begin{cases} \frac{A}{\pi(1-n^2)} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Fourier transform:-

Fourier transform gives the frequence information of the periodic signal

Fourier Transform of the signal $x(t)$

$$x(\omega) = FT[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

time → forward basis

ω is a continuous variable

→ Inverse fourier transform of $x(\omega)$

$$x(t) = IFT[X(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(\omega)\, e^{j\omega t}\, d\omega$$

short hand notation:-

$$\underbrace{x(t)}_{\text{time domain}} \xleftrightarrow{FT} \overbrace{x(\omega)}^{\text{frequency domain}}$$

Read as $x(t)$, $x(\omega)$ forms Fourier Transform

where $x(t) = IFT[x(\omega)]$

$x(\omega) = FT[x(t)]$

Dirichlet's conditions:

① signal must be absolutely integrable

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$$

(or)

Energy must be finite

$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt < \infty$$

② signal must be finite no. of discontinuties in the finite interval

③ signal must have finite no. of maxima's and minima's in the finite interval

⇒ conjugate symmetry property

$$x(\omega) = FS[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$x^*(\omega) = \left[\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt\right]^*$$

$$= \int_{-\infty}^{\infty} \left(x(t)\, e^{-j\omega t}\right)^* dt$$

$$= \int_{-\infty}^{\infty} x^*(t)\left(e^{-j\omega t}\right)^* dt$$

$$x^*(\omega) = \int_{-\infty}^{\infty} x^*(t) e^{j\omega_0 t} dt$$

If $x(t)$ is periodic signal

Then, $x(t) = x^*(t)$

$$x^*(\omega) = \int_{-\infty}^{\infty} x(t) e^{j\omega t} dt$$

$$x^*(-\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$x^*(-\omega) = x(\omega)$$

$\therefore$ If $x(t)$ is real term

$$\boxed{x(\omega) = x^*(-\omega)}$$

If $x(t)$ is real value signal then its F.T possess conjugate symmetry

$x(\omega) = x^*(-\omega)$

$x(\omega)$ is always complex

$x(\omega) = |x(\omega)| e^{j\theta(\omega)}$

where $\theta(\omega) = \angle x(\omega)$

$x^*(-\omega)$ is also complex

$x^*(-\omega) = |x^*(-\omega)| e^{-j\theta(-\omega)} = |x(-\omega)| e^{-j\theta(-\omega)}$

where $\theta(\omega) = \angle x^*(\omega)$

$|x(\omega)| e^{j\theta\omega} = |x(-\omega)| e^{-j\theta(-\omega)}$

Magnitude function $|x(\omega)| = |x(-\omega)|$

phase function

$\theta(\omega) = -\theta(-\omega)$

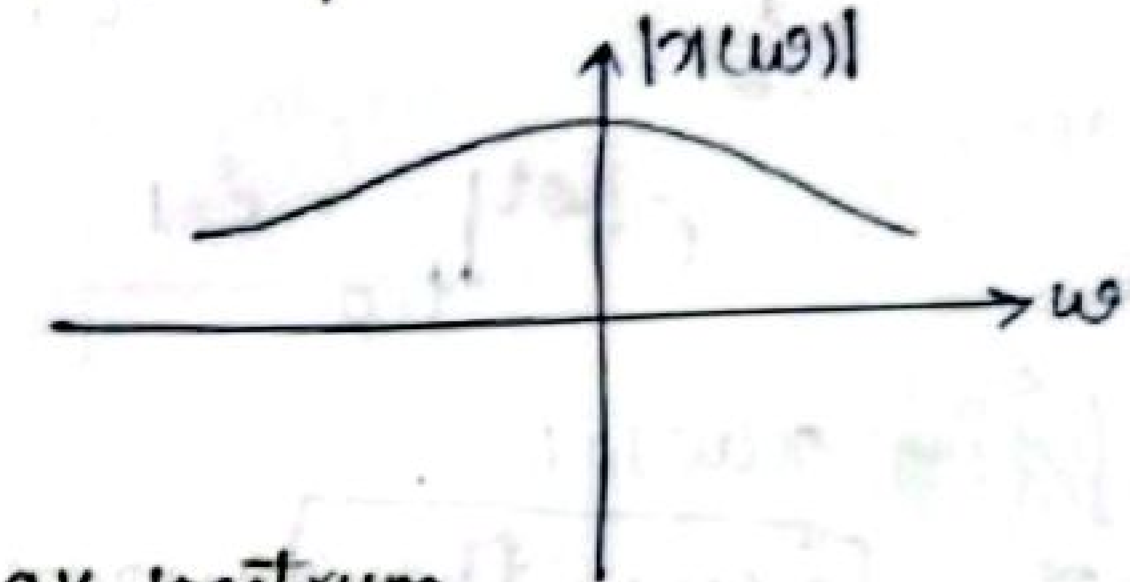
magnitude function is even

phase-function is odd

Magnitude spectrum

$\rightarrow$ The plot b/w $|x(\omega)|$ vs ω is called magnitude spectrum

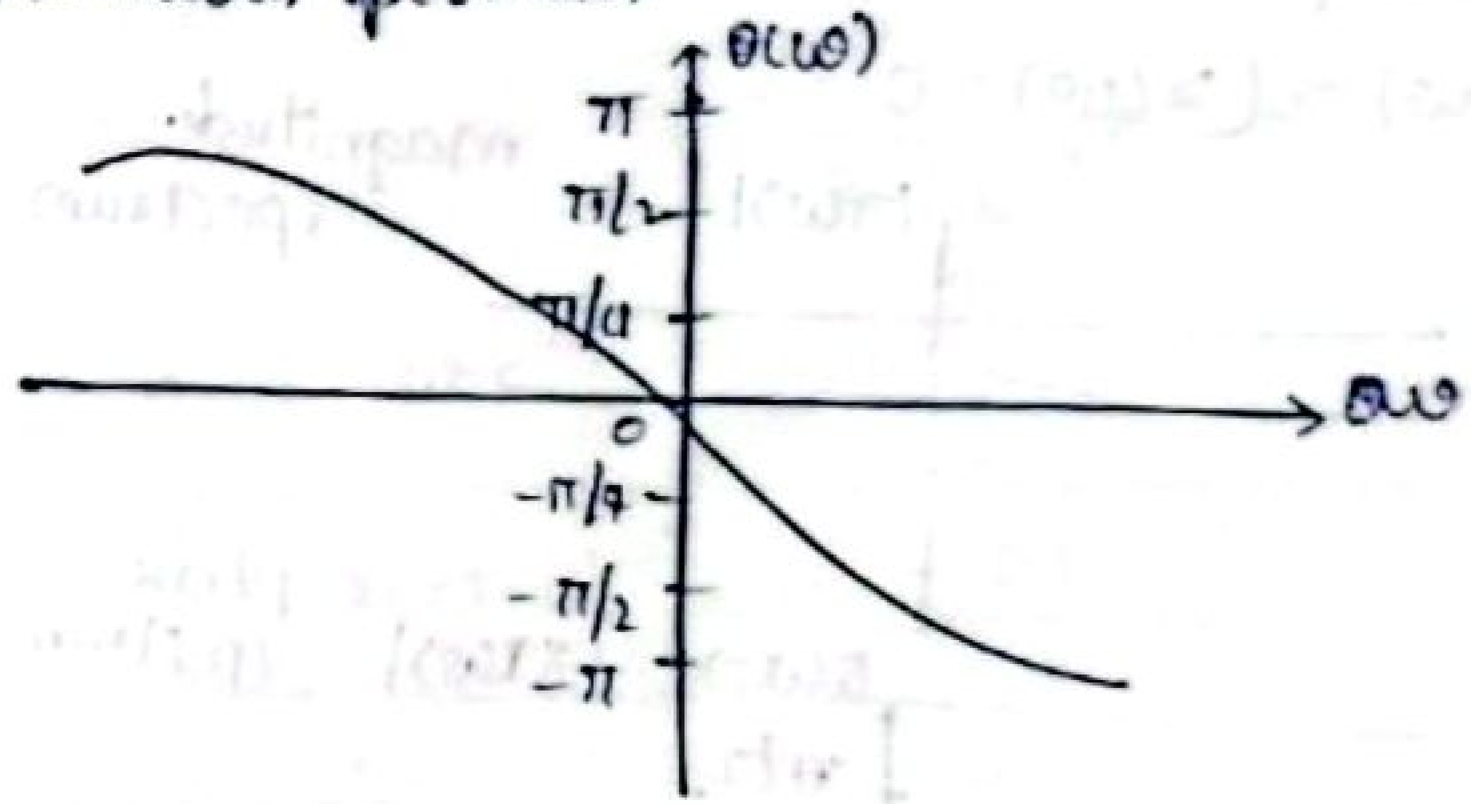
$\rightarrow$ Magnitude spectrum posses even symmetry and continuous spectrum



$\rightarrow$ phase spectrum

$\rightarrow$ The plot b/w $\theta(\omega)$ vs ω is called phase spectrum

$\rightarrow$ phase spectrum posses odd symmetry and continuous spectrum



The combination of M.S and P.S is called frequency function

① find Fourier transforms of $x(t)$

$$x(\omega) = FT[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt$$

sampling property of an impluse

$$= \int_{-\infty}^{\infty} \delta(t) f(t)\, dt = f(t)\Big|_{0=t}$$

$$\because e^{-j\omega t}\Big|_{t=0} = e^{0} = 1$$

$x(\omega) = 1$

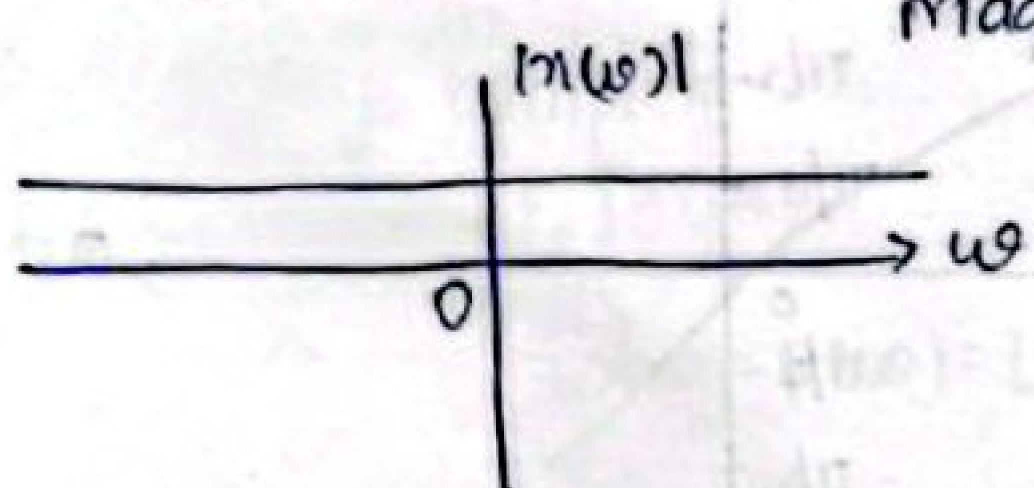
Fourier transforms of impluse function: $\boxed{\because \delta(t) \xleftrightarrow{FT} 1}$ (or) $IFT[1] = \delta(t)$

$\Leftrightarrow x(\omega) = 1$

$|x(\omega)| = 1$

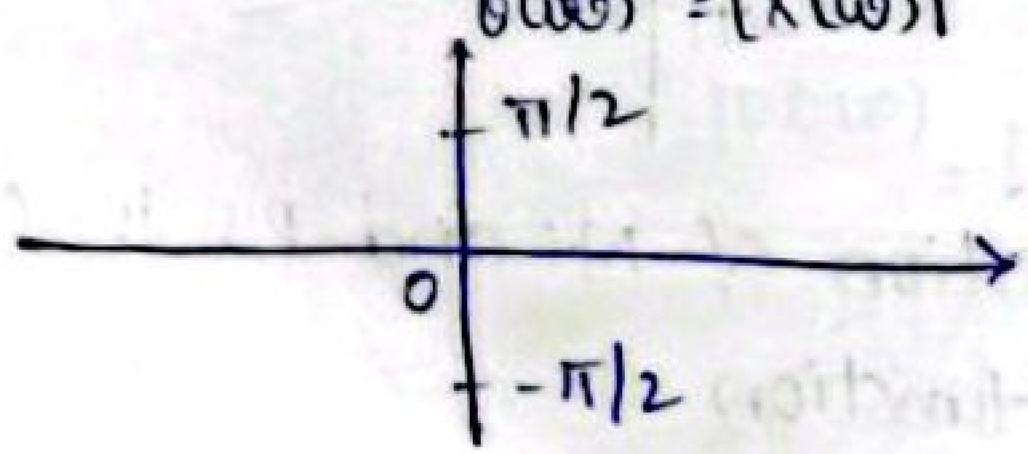
$\theta(\omega) = \angle x(\omega) = 0$

Magnitude spectrum



phase spectrum

$\theta(\omega) = \angle X(\omega)$

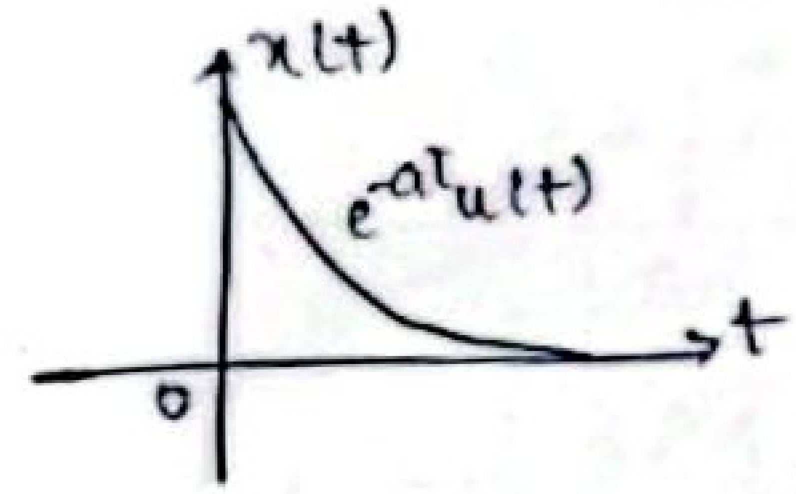


② find Fourier ...

where $a>0$

$$x(\omega) = FT[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-at} u(t)\, e^{-j\omega t}\, dt$$



$$= \int_{-\infty}^{0} u(t) e^{-at} e^{-j\omega t} + \int_{0}^{\infty} e^{-at} \cdot e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt$$

$$= \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_{0}^{\infty}$$

$$= \frac{-1}{a+j\omega}\left[e^{-\infty} - e^{0}\right] = \frac{1}{a+j\omega}$$

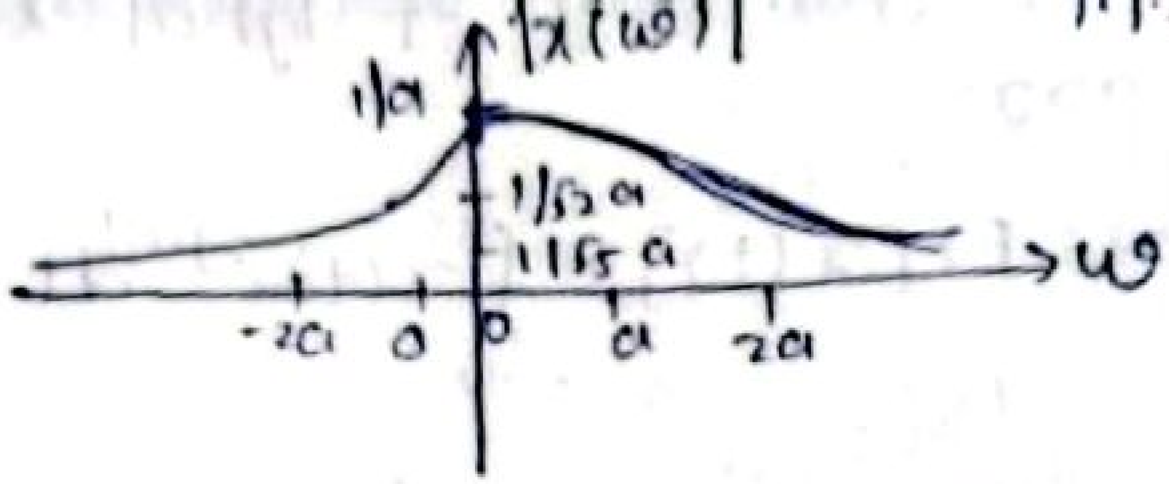
$$\therefore x(\omega) = \frac{1}{a+j\omega}$$

$$\boxed{\because x(t) = e^{-at} u(t) \xleftrightarrow{FT} x(\omega) = \frac{1}{a+j\omega}}$$

$x(\omega) = \frac{1}{a+j\omega}$

$|x(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$

$\theta(\omega) = \angle x(\omega) = 0 - \tan^{-1}\left(\frac{\omega}{a}\right)$



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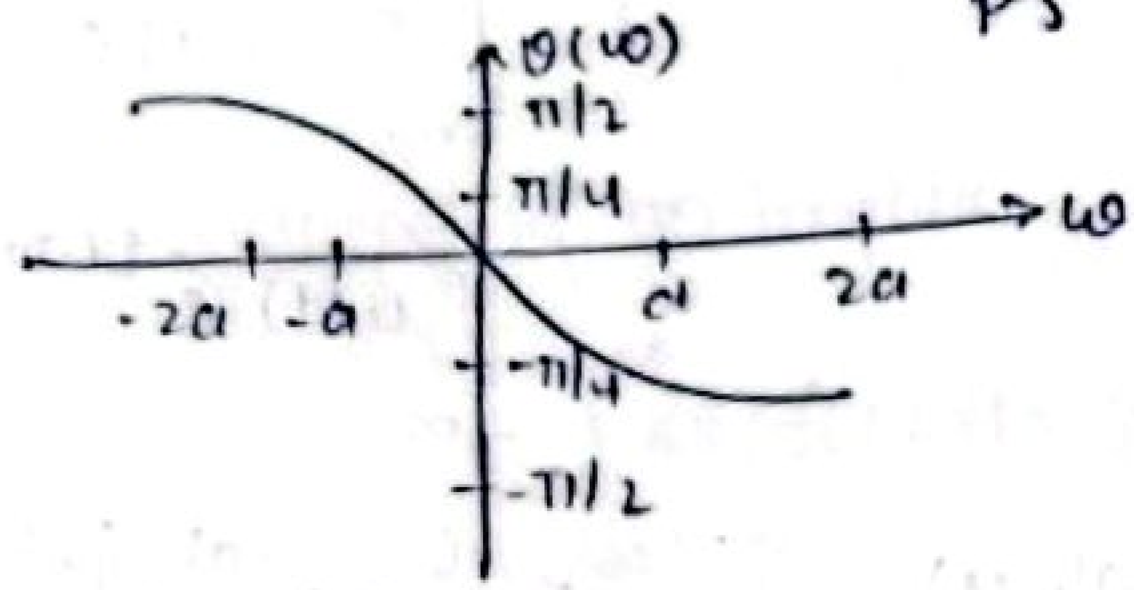


Fular's Theorem:-

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} x(t)[\cos(\omega t) - j\sin(\omega t)]\, dt$$

$$X(\omega) = \underbrace{\int_{-\infty}^{\infty} \underset{e}{x(t)} \underset{e}{\cos(\omega t)} dt}_{e} - \underbrace{\int_{-\infty}^{\infty} \underset{e}{x(t)}\, j\underset{o}{\sin(\omega t)} dt}_{0}$$

case ①:-

$x(t)$ is even

$$X(\omega) = 2\int_{0}^{\infty} x(t)\cos(\omega t)\, dt$$

if $f(x)$ is even

$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

otherwise '0'

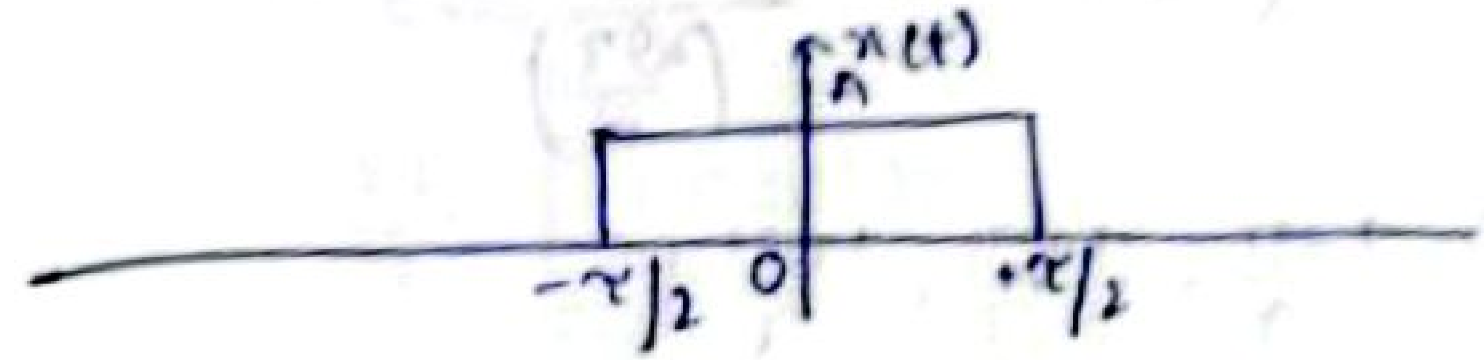
case ②:-

$x(t)$ is odd

$$X(\omega) = -2j\int_{0}^{\infty} x(t)\sin(\omega t)\,dt$$

Q) Find the fourier transform of rectangle pulse.

$x(t) = A\,\text{rect}\left(\frac{t}{\tau}\right)$



This is even signal

$\therefore$ Fourier Transform is

$$X(\omega) = 2\int_{0}^{\infty} x(t)\cos(\omega t)\,dt$$

$$= 2\int_{0}^{\tau/2} A\,\text{rect}\left(\frac{t}{\tau}\right) \cos(\omega t)\, dt + \int_{\tau/2}^{\infty} 0$$

$$= 2A \int_{0}^{\tau/2} \cancel{\text{rect}\left(\frac{t}{\tau}\right) dt}$$

$$= 2A\int_{0}^{\tau/2} \cos(\omega t)\,dt$$

$$= 2A\left(\frac{\sin(\omega t)}{\omega}\right)_{0}^{\tau/2}$$

$$= \frac{2A}{\omega}\left[\sin\left(\omega \cdot \frac{\tau}{2}\right) - \sin(0)\right]$$

$$= \frac{2A}{\omega}\sin\left(\frac{\omega\tau}{2}\right)$$

$$= \cancel{2}A\,\frac{\sin\left(\frac{\omega\tau}{2}\right)}{\omega\left(\frac{\tau}{2}\right)} \times \frac{\tau}{\cancel{2}}$$

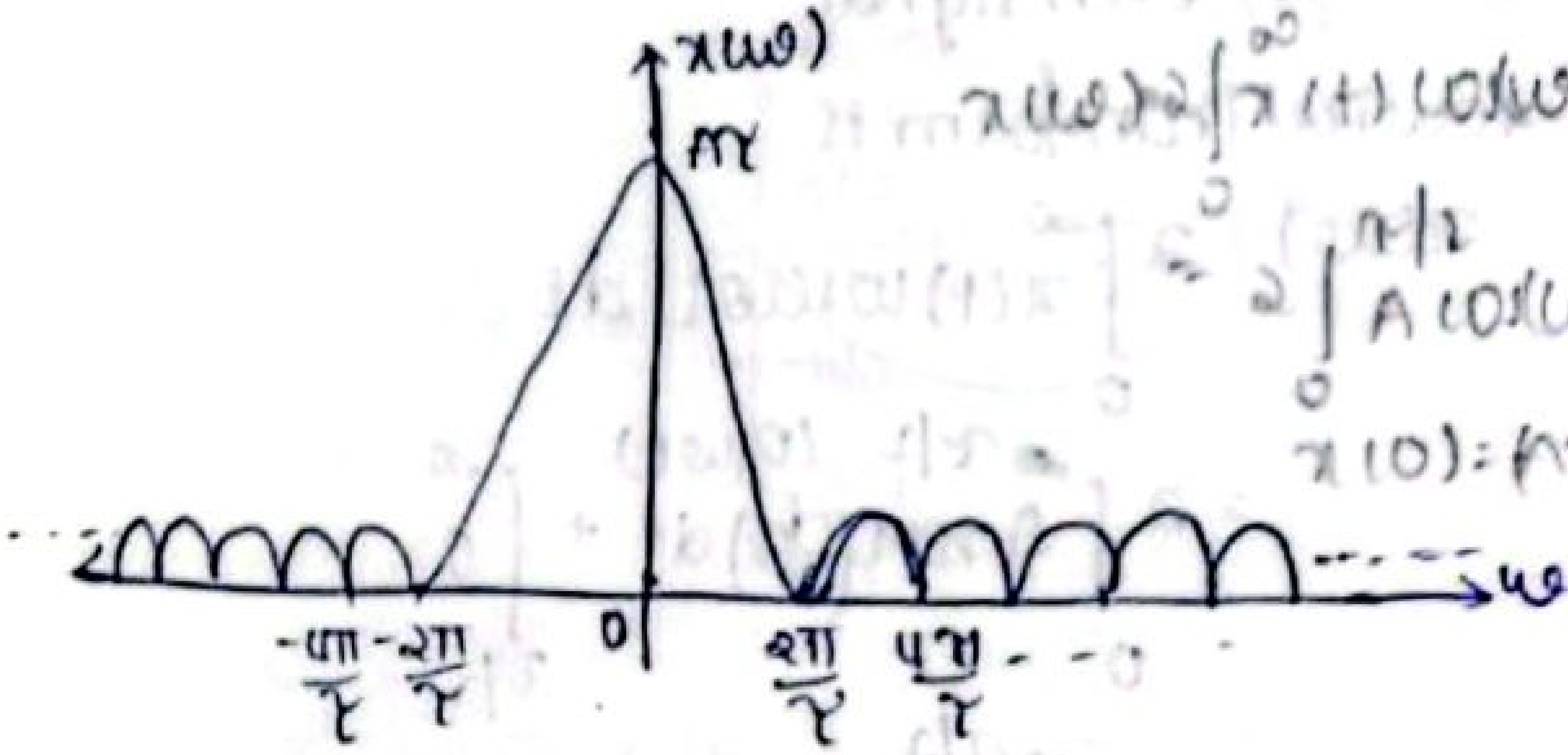
$$X(\omega) = A\tau\, Sa\left(\frac{\omega\tau}{2}\right)$$

$$FT\left[A\,\text{rect}\left(\frac{t}{\tau}\right)\right] = A\tau\, Sa\left(\frac{\omega\tau}{2}\right)$$

$$\boxed{A\,\text{rect}\left(\frac{t}{\tau}\right) \xleftrightarrow{FT} A\tau\, Sa\left(\frac{\omega\tau}{2}\right)}$$

$x(\omega) = A\tau\, Sa\left(\frac{\omega\tau}{2}\right) = A\tau \frac{\sin(\omega\tau/2)}{\left(\frac{\omega\tau}{2}\right)}$

at $\omega = \pm\frac{2\pi}{\tau}, \pm\frac{4\pi}{\tau}, \pm\frac{6\pi}{\tau}, \ldots\ldots$



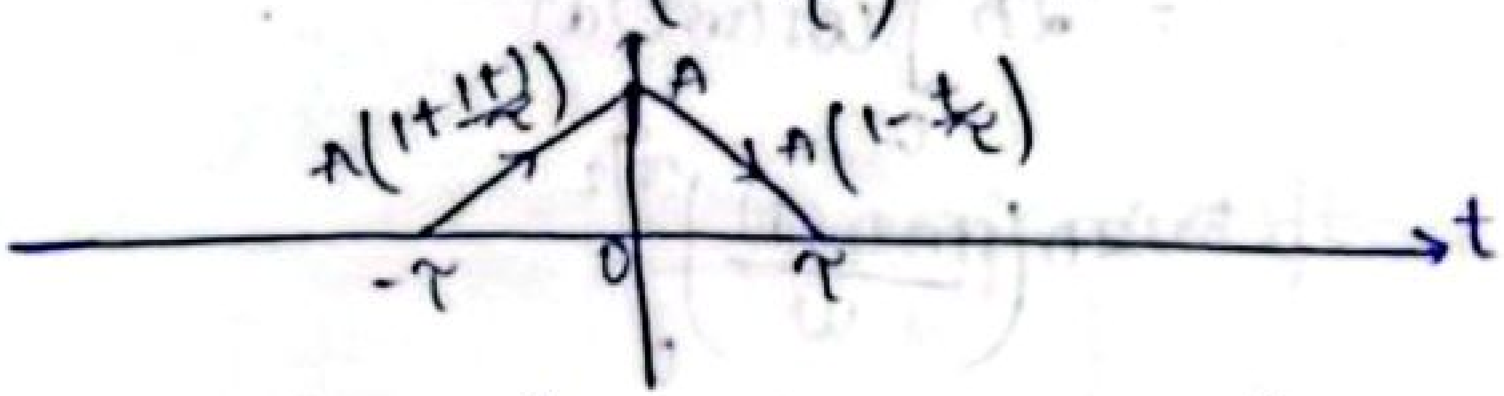
$x(\omega) = 2\int_0^\infty x(t)\cos(\omega t)\,dt$

$= 2\int_0^{\tau/2} A\cos(\omega t)\,dt$

$x(0) = A\tau$

(Q) Find the fourier Transform of triangle pluse

$x(t) = A\left(1 - \frac{|t|}{\tau}\right)$



sine waveform is symmetric about y axis

it is even signal

FT: $x(\omega) = 2\int_0^\infty x(t)\cos(\omega t)\,dt$

$= 2\int_0^{\tau} A\left(1-\frac{t}{\tau}\right)\cos(\omega t)\,dt + 0$

$= 2A\int_0^{\tau}\left(1-\frac{t}{\tau}\right)\cos(\omega t)\,dt$

$= 2A\left\{\left[\left(1-\frac{t}{\tau}\right)\frac{\sin(\omega t)}{\omega}\right]_0^\tau - \int_0^\tau \frac{\sin(\omega t)}{\omega}\left(0-\frac{1}{\tau}\right)dt\right.$

$= 2A\left[0 + \int_0^\tau \frac{\sin(\omega t)}{\omega}\,dt\ \frac{1}{\tau}\right]$

$= \frac{2A}{\tau}\int_0^\tau \sin(\omega t)\,dt$

$= \frac{2A}{\omega\tau}\left[-\frac{\cos(\omega t)}{\omega}\right]_0^\tau$

$= -\frac{2A}{\omega^2\tau}\left[\cos(\omega\tau) - \cos 0\right]$

$= +\frac{2A}{\omega^2\tau}\left[1 - \cos(\omega\tau)\right]$ $\boxed{1-\cos\theta = 2\sin^2\frac{\theta}{2}}$

$= \frac{2A}{\omega^2\tau}\, 2\sin^2\left(\frac{\omega\tau}{2}\right)$

$= \frac{4A}{\omega^2\tau}\sin^2\left(\frac{\omega\tau}{2}\right)$

$= 4A\,\frac{\sin^2\left(\frac{\omega\tau}{2}\right)}{\frac{\omega^2\tau^2}{2}}\cdot\frac{\tau}{2}$

$= 2A\tau\,\frac{\sin^2\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)^2}$

$= A\tau\left(\frac{\sin\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)}\right)^2$

$= A\tau\, Sa^2\left(\frac{\omega\tau}{2}\right)$

$FT\left[A\left[1-\frac{|t|}{\tau}\right]\right] = A\tau\, Sa^2\left(\frac{\omega\tau}{2}\right)$

$$\boxed{A\left[1-\frac{|t|}{\tau}\right] \xleftrightarrow{FT} A\tau\, Sa^2\left(\frac{\omega\tau}{2}\right)}$$

Properties of Fourier Transform

① Timescaling properties:-

If $x(t) \xleftrightarrow{FT} x(\omega)$

Then $x(at) \xleftrightarrow{FT} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$

<u>proof:-</u>

case ①:- $a>0$

$$FT[x(at)] = \int_{-\infty}^{\infty} x(at) e^{-j\omega t} dt$$

let $at = t_1 \Rightarrow t = \frac{t_1}{a}$

$t \to -\infty \quad t_1 \to -\infty$

$t \to \infty \quad t_1 \to \infty$

$$F[x(at)] = \frac{1}{a} \int_{-\infty}^{\infty} x(t_1) e^{-j\omega \frac{t_1}{a}} dt_1$$

$t_1 \to t$

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(t) e^{-j\left(\frac{\omega}{a}\right)t} dt$$

$$= \frac{1}{a} X\left(\frac{\omega}{a}\right)$$

$\therefore F[x(at)] = \frac{1}{a} X\left(\frac{\omega}{a}\right)$.

$\therefore$ case ②:- $a<0$

$$F[x[at]] = \frac{1}{a} X\left(\frac{-\omega}{a}\right)$$

② Time reversal property:-

If $x(t) \xleftrightarrow{FT} x(\omega)$

Then $x(-t) \xleftrightarrow{FT} x(-\omega)$

$$FT[x(bt)] = \int_{-\infty}^{\infty} x(bt) e^{-j\omega t} dt$$

Here $a = -1$

$$\therefore FT[x(-t)] = \int_{-\infty}^{\infty} x(-t) e^{-j\omega t} dt$$

$\therefore x(at) \xleftrightarrow{FT} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$

$x(-t) \xleftrightarrow{FT} \frac{1}{|-1|} X\left(\frac{\omega}{-1}\right)$

$x(-t) \xleftrightarrow{FT} x(-\omega)$

$\therefore FT[x(-t)] = X(-\omega)$

<u>Example:-</u>

$x(t) = e^{+at} u(-t)$

we know that

$x(t) = e^{-at} u(-t) \xleftrightarrow{FT} x(\omega) = \frac{1}{a+j\omega}$

$\therefore$ Apply time reversal property

$\therefore e^{+at} u(-t) = x(-t)$ (t replace with $-t$)

$$\boxed{e^{at} u(-t) \xleftrightarrow{FT} \frac{1}{a-j\omega}}$$

③ linearity property:-

If $x_1(t) \xleftrightarrow{FT} x_1(\omega)$

$x_2(t) \xleftrightarrow{FT} x_2(\omega)$

Then $a_1 x_1(t) + a_2 x_2(t) \xleftrightarrow{FT} a_1 X_1(\omega) + a_2 X_2(\omega)$

proof:

$$FT[a_1x_1(t)+a_2x_2(t)] = \int_{-\infty}^{\infty}[a_1x_1(t)+a_2x_2(t)]e^{-j\omega t}dt$$

$$= a_1\int_{-\infty}^{\infty}x_1(t)e^{-j\omega t}dt + a_2\int_{-\infty}^{\infty}x_2(t)e^{-j\omega t}dt$$

$$FT[a_1x_1(t)+a_2x_2(t)] = a_1 FT[x_1(t)] + a_2 FT[x_2(t)]$$

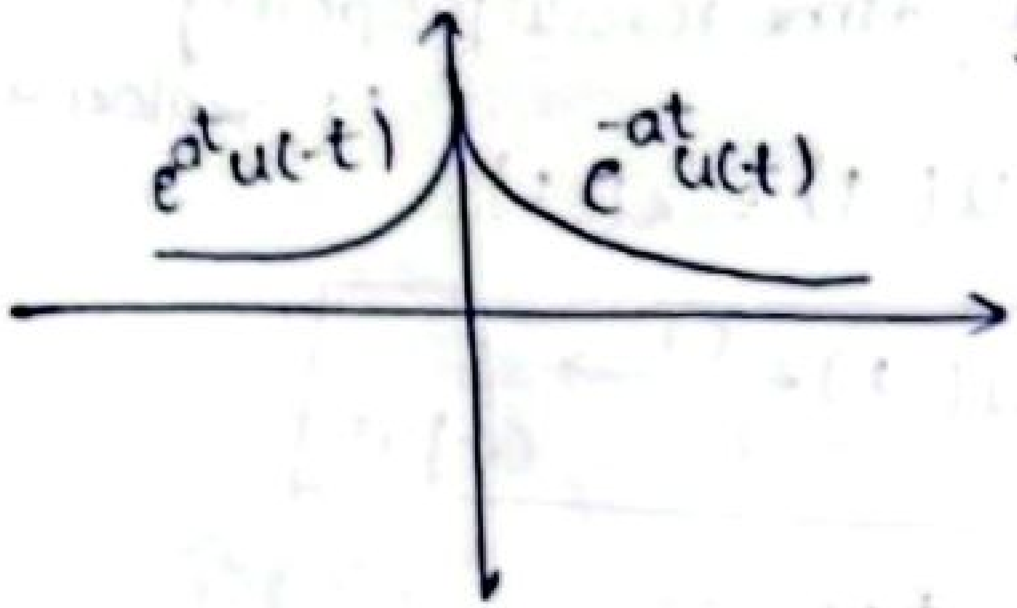
$$FT[a_1x_1(t)+a_2x_2(t)] = a_1X_1(\omega)+a_2X_2(\omega)$$

$$\therefore \boxed{a_1x_1(t)+a_2x_2(t) \xleftrightarrow{FT} a_1X_1(\omega)+a_2X_2(\omega)}$$

(5) Find the FT of signal $x(t)=e^{-a|t|}$

$$x(t) = \begin{cases} e^{-at} & \text{for } t>0 \\ e^{at} & \text{for } t<0 \end{cases}$$

$$\therefore x(t) = e^{-at}u(t) + e^{at}u(-t)$$



This is two sided exponentially decaying signal

$$e^{-a|t|} = e^{-at}u(t) + e^{at}u(-t)$$

Apply linearity property

$$\therefore FT[e^{-a|t|}] = FT[e^{-at}u(t)] + FT[e^{at}u(-t)]$$

$$= \frac{1}{a+j\omega} + \frac{1}{a-j\omega}$$

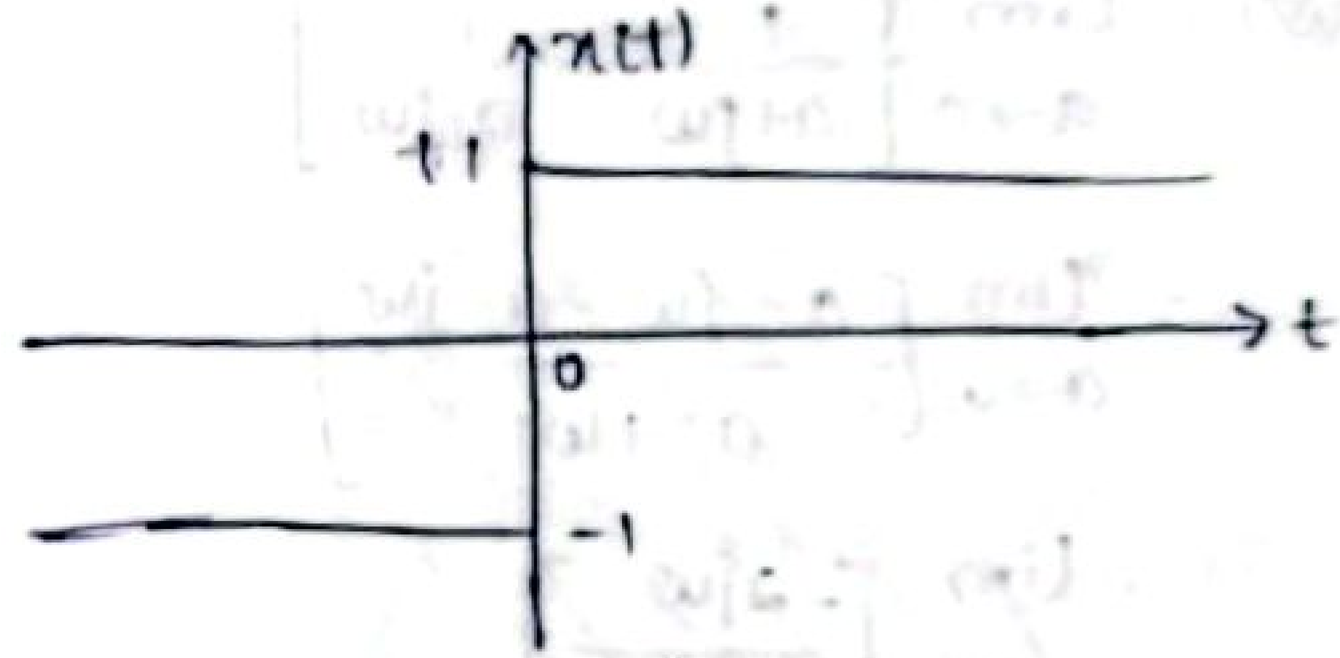
$$= \frac{a-j\omega \cdots}{a^2+\omega^2}$$

$$FT[e^{-a|t|}] = \frac{2a}{a^2+\omega^2}$$

$$\therefore \boxed{e^{-a|t|} \xleftrightarrow{FT} \frac{2a}{a^2+\omega^2}}$$

(6) Find the FT of the signum function?

$$x(t) = sgn(t)$$

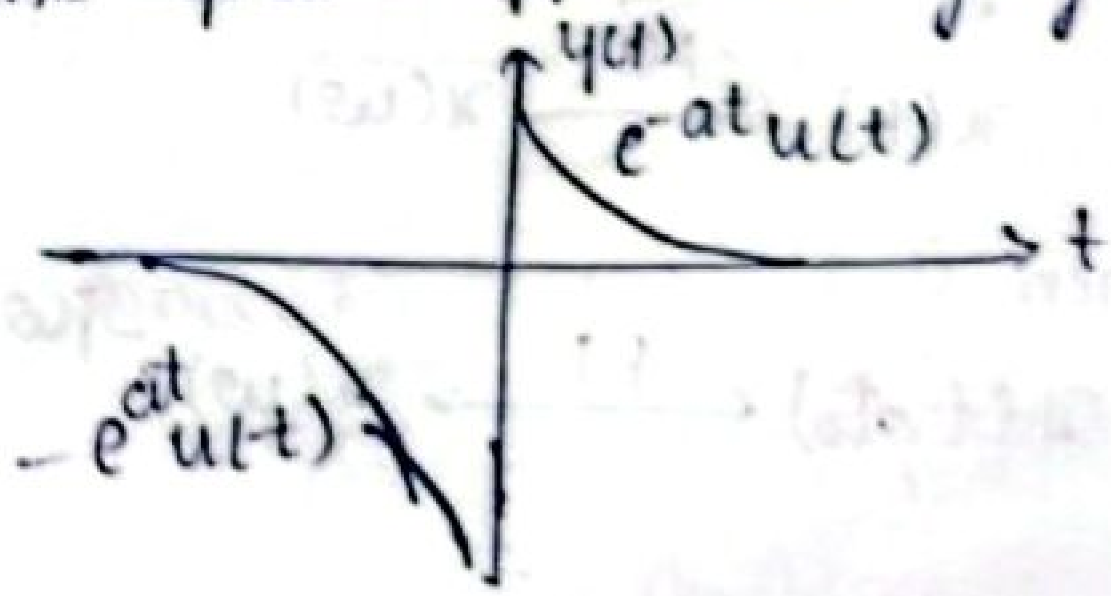


Absolute integrability

$$\int_{-\infty}^{\infty}|x(t)|dt = \int_{-\infty}^{\infty}1\,dt = \infty$$

The signal is not absolute integrable ∴ we can't find Fourier Transformer in direct manner. However, we can find FT of this signal in a limiting form.

This signal is approximating by



$y(t) = e^{-at}u(t) - e^{at}u(-t)$

$\therefore \lim_{a\to 0} y(t) = x(t)$

when $x(t) = \lim_{a\to 0}\left[e^{-at}u(t) - e^{at}u(-t)\right]$

Apply linearity property

$FT[x(t)] = \lim_{a\to 0}\left\{FT[e^{-at}u(t)] - FT[e^{at}u(-t)]\right\}$

$x(\omega) = \lim_{a\to 0}\left[\frac{1}{a+j\omega} - \frac{1}{a-j\omega}\right]$

$= \lim_{a\to 0}\left[\cancel{\frac{a-j\omega-a-j\omega}{a^2+\omega^2}}\right]$

$= \lim_{a\to 0}\left[\cancel{\frac{-2j\omega}{a^2+\omega^2}}\right]$

$\therefore \frac{1}{j\omega} + \frac{1}{j\omega}$

$= \frac{2}{j\omega}$

$$\boxed{sgn[t] \xleftrightarrow{FT} \frac{2}{j\omega}}$$

① Time shifting property:

If $x(t) \xleftrightarrow{FT} x(\omega)$

Then

$x(t-t_0) \xleftrightarrow{FT} x(\omega)e^{-j\omega t_0}$

proof:

$FT[x(t-t_0)] = \int_{-\infty}^{\infty} x(t-t_0)e^{-j\omega t}dt$

let $t - t_0 = t_1$

$t \to -\infty,\ t_1 \to -\infty$

$t \to \infty,\ t_1 \to \infty$

$dt = dt_1$

$\therefore \int_{-\infty}^{\infty} x(t_1)e^{-j\omega(t_1+t_0)}dt_1$

$= \int_{-\infty}^{\infty} x(t_1)e^{-j\omega t_1}\cdot e^{-j\omega t_0}dt_1$

$= e^{-j\omega t_0}\int_{-\infty}^{\infty} x(t_1)e^{-j\omega t_1}dt_1$

$t_1 \leftrightarrow t$

$= e^{-j\omega t_0}\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$

$= e^{-j\omega t_0}\,x(\omega)$

$$\boxed{\therefore x(t-t_0) \xleftrightarrow{FT} x(\omega)e^{-j\omega t_0}}$$

similary

$$\boxed{x(t+t_0) \xleftrightarrow{FT} x(\omega)e^{j\omega t_0}}$$

Example: find FT $x(t) = \delta(t-1)$

$y(t) = \delta(t) \xleftrightarrow{FT} y(\omega) = 1$

Apply Time shifting property on $y(t)$

by an amount of one

$$y(t-1) \xleftrightarrow{FT} y(\omega)e^{-j\omega(1)}$$

$$\boxed{\delta(t-1) \xleftrightarrow{FT} (1)e^{-j\omega}}$$

similarly

$$\boxed{\delta(t+1) \xleftrightarrow{FT} e^{j\omega}}$$

(8) Find the FT of signal $x(t) = e^{-a|t-t_0|}$

let $y(t) = e^{-a|t|} \xleftrightarrow{FT} y(\omega) = \frac{2a}{a^2+\omega^2}$

Apply time shifting property

$$y(t-t_0) \xleftrightarrow{FT} y(\omega)e^{-j\omega t_0}$$

$$\boxed{e^{-a|t-t_0|} \xleftrightarrow{FT} \left(\frac{2a}{a^2+\omega^2}\right)e^{-j\omega t_0}}$$

(5) Duality property:

If $x(t) \xleftrightarrow{FT} X(\omega)$

then $X(t) \xleftrightarrow{FT} 2\pi x(-\omega)$

proof:

$$x(t) = IFT[X(\omega)]$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{+j\omega t}\,d\omega$$

let $t \to -t$

$$x(-t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{-j\omega t}\,d\omega$$

Replace t with ω with t

$$x(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(t)e^{-j\omega t}\,dt$$

$$2\pi x(-\omega) = \int_{-\infty}^{\infty} X(t)e^{-j\omega t}\,dt$$

$$2\pi x(-\omega) = FT[X(t)]$$

$$\boxed{X(t) \xleftrightarrow{FT} 2\pi x(-\omega)}$$

	$x(t)$	$X(\omega)$
(1)	$\delta(t)$	1
(2)	$e^{-at}u(t)$	$\frac{1}{a+j\omega}$
(3)	$e^{at}u(-t)$	$\frac{1}{a-j\omega}$
(4)	$A\,\mathrm{rect}(t/\tau)$	$A\tau\, sa\left(\frac{\omega\tau}{2}\right)$
(5)	$A\left(1-\frac{\lvert t\rvert}{\tau}\right)$	$A\tau\, sa^2\left(\frac{\omega\tau}{2}\right)$
(6)	$e^{-a\lvert t\rvert}$	$\frac{2a}{a^2+\omega^2}$
(7)	$\mathrm{sgn}(t)$	$\frac{2}{j\omega}$
(8)	$sa(Bt)$	$\frac{\pi}{B}\mathrm{rect}\left(\frac{\omega}{2B}\right)$
(9)	$sa^2(Bt)$	$\frac{\pi}{B}\left(1-\frac{\lvert\omega\rvert}{2B}\right)$
(10)	1	$2\pi\delta(\omega)$

Ex:- $x(t) = sa(Bt)$

$$y(A) = A\,rect(t/\tau) \xleftrightarrow{FT} Y(\omega) = A\tau\, sa\left(\frac{\omega\tau}{2}\right)$$

Apply duality property on $y(t)$

$$y(t) \xleftrightarrow{FT} 2\pi\, y(-\omega)$$

∵ replace ω with t

$$y(t) = A\tau\, sa\left(\frac{t\tau}{2}\right)$$

$$\because A\tau\, sa\left(\frac{t\tau}{2}\right) \xleftrightarrow{FT} 2\pi A\, rect\left(\frac{-\omega}{\tau}\right)$$

let $\tau/2 = B \Rightarrow \tau = 2B$

$A = 1$

$$(1)\, 2B\, sa(Bt) \xleftrightarrow{FT} 2\pi(1)\, rect\left(\frac{-\omega}{2B}\right)$$

$$\because 2B\, sa(Bt) \xleftrightarrow{FT} 2\pi\, rect\left(\frac{-\omega}{2B}\right)$$

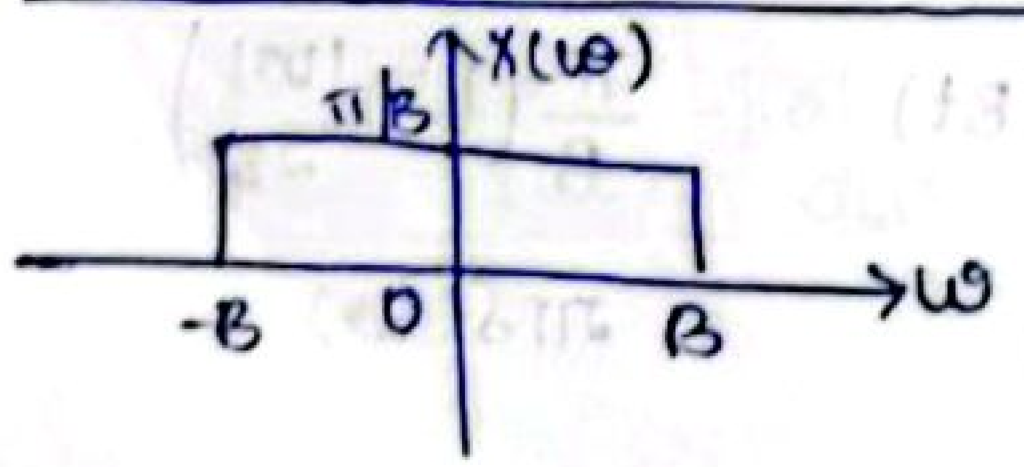
Rectangle function is a even function

$$\because 2B\, sa(Bt) \xleftrightarrow{FT} 2\pi\, rect\left(\frac{\omega}{2B}\right)$$

Amplitude scaling by a factor $1/2B$

$$\frac{2B}{2B}\, sa(Bt) \xleftrightarrow{FT} \frac{2\pi}{2B}\, rect\left(\frac{\omega}{2B}\right)$$

$$\boxed{sa(Bt) \xleftrightarrow{FT} \frac{\pi}{B}\, rect\left(\frac{\omega}{2B}\right)}$$



(b) find the F.T of signal $sa^2(Bt)$

let $y(A) = A\left(1 - \frac{|t|}{\tau}\right) \xleftrightarrow{FT} Y(\omega) = A\tau\, sa^2\left(\frac{\omega\tau}{2}\right)$

Apply duality property on $y(t)$

$$y(t) \xleftrightarrow{FT} 2\pi\, y(-\omega)$$

replace ω with t

$$y(t) = A\tau\, sa^2\left(\frac{t\tau}{2}\right)$$

$$\because A\tau\, sa^2\left(\frac{t\tau}{2}\right) \xleftrightarrow{FT} 2\pi A\left(1 - \frac{|\omega|}{\tau}\right)$$

let $A = 1$, $\tau = 2B$

$$(1)\, 2B\, sa^2(Bt) \xleftrightarrow{FT} 2\pi(1)\left(1 - \frac{|\omega|}{2B}\right)$$

Amplitude scaling by $2B$

$$\boxed{sa^2(Bt) \xleftrightarrow{FT} \frac{\pi}{B}\left(1 - \frac{|\omega|}{2B}\right)}$$

(c) find the FF of signal $x(t) = 1$

let $y(t) = \delta(t) \xleftrightarrow{FT} 1 = Y(\omega)$

Apply duality propert on $y(t)$

$$y(t) \xleftrightarrow{FT} 2\pi\, y(-\omega)$$

replace ω with t

$$y(t) = 1 \xleftrightarrow{FT} \underbrace{2\pi\, \delta(-\omega)}_{\text{Even function}}$$

$$\boxed{1 \xleftrightarrow{FT} 2\pi\, \delta(\omega)}$$

(6) Differentiation in time domain property

If $x(t) \xleftrightarrow{FT} X(\omega)$

Then $\frac{dx(t)}{dt} \xleftrightarrow{FT} j\omega X(\omega)$

proof:

~~x(t) ↔ IFT~~

$x(t) = IFT[X(\omega)]$

$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$

differentiate this with d/dt

$$\frac{dx(t)}{dt} = \frac{1}{2\pi} \frac{d}{dt}\left[\int_{-\infty}^{\infty} X(\omega) e^{+j\omega t} d\omega\right]$$

$$\frac{dx(t)}{dt} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \frac{d}{dt} e^{j\omega t} d\omega$$

$$\frac{dx(t)}{dt} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} (j\omega) d\omega$$

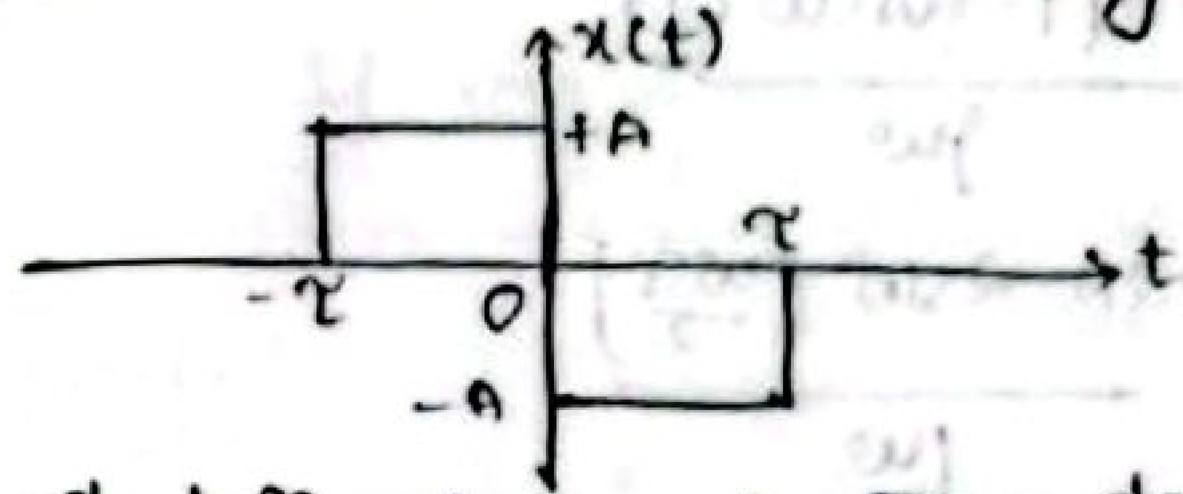
$$\frac{dx(t)}{dt} = \frac{1}{2\pi} \int_{-\infty}^{\infty} [j\omega X(\omega)] e^{j\omega t} d\omega$$

$$\frac{dx(t)}{dt} = IFT[j\omega X(\omega)]$$

$$FT\left[\frac{d(x(t))}{dt}\right] = j\omega X(\omega)$$

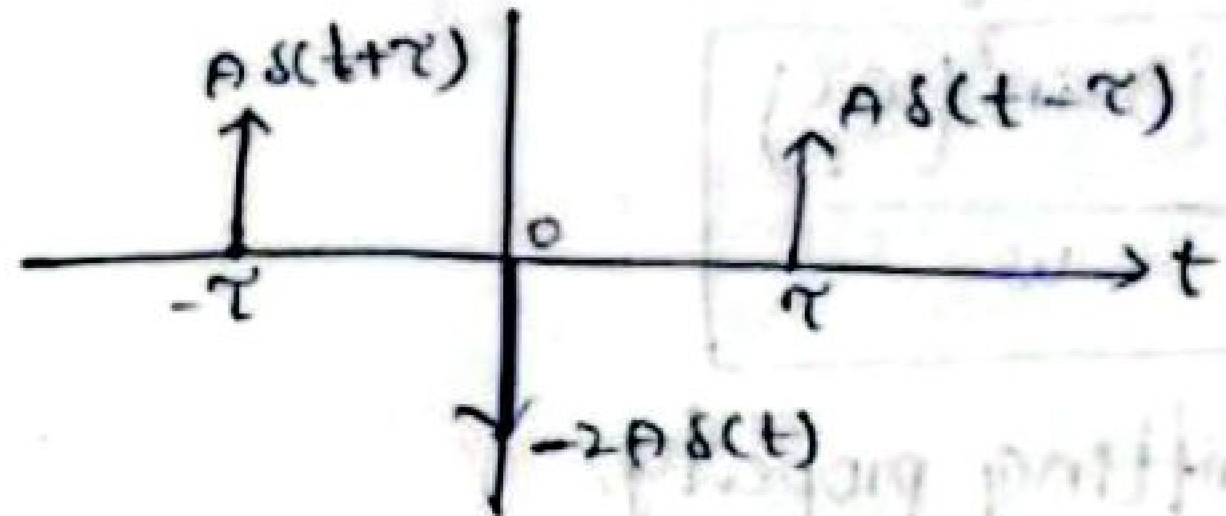
$$\boxed{\frac{dx(t)}{dt} \xleftrightarrow{FT} j\omega X(\omega)}$$

(8) Find the FT in the following signal



using differentiation in Time domain property

$\frac{dx(t)}{dt}$



$$\frac{dx(t)}{dt} = A\delta(t+\tau) - 2A\delta(t) + A\delta(t-\tau)$$

Apply FT on Both

$$FT\left[\frac{dx(t)}{dt}\right] = FT\left[A\delta(t+\tau) - 2A\delta(t) + A\delta(t-\tau)\right]$$

$$FT\left[\frac{dx(t)}{dt}\right] = A\,FT[\delta(t+\tau)] - 2A\,FT[\delta(t)] + A\,FT[\delta(t-\tau)]$$

$$FT\left[\frac{dx(t)}{dt}\right] = A X(\omega) e^{j\omega\tau} - 2A(1) + A X(\omega) e^{-j\omega\tau}$$

$$j\omega X(\omega) = A(1) e^{j\omega\tau} - 2A + A(1) e^{-j\omega t}$$

$$j\omega X(\omega) = A\left[e^{j\omega t} + e^{-j\omega t}\right] - 2A$$

$$\cancel{j\omega} X(\omega) = \frac{2A\cos(\omega\tau) - 2A}{j\omega}$$

$$X(\omega) = \frac{2A[1-\cos(\omega\tau)]}{j\omega}$$

$$X(\omega) = \frac{-2A \cdot 2\sin^2\left(\frac{\omega\tau}{2}\right)}{j\omega}$$

$$X(\omega) = \frac{-4A\sin^2\left(\frac{\omega\tau}{2}\right)}{j\omega}$$

$$\boxed{X(\omega) = \frac{4jA\sin^2\left(\frac{\omega\tau}{2}\right)}{\omega}}$$

⑦ Frequency shifting property:-

If $x(t) \xleftrightarrow{FT} X(\omega)$

then $x(t)e^{j\omega_0 t} \xleftrightarrow{FT} X(\omega-\omega_0)$

proof:-

$$FT[x(t)e^{j\omega_0 t}] = \int_{-\infty}^{\infty}[x(t)e^{j\omega_0 t}]e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} x(t)e^{-j(\omega-\omega_0)t}dt$$

$$FT[x(t)e^{j\omega_0 t}] = X(\omega-\omega_0)$$

$$x(t)e^{j\omega_0 t} \xleftrightarrow{FT} X(\omega-\omega_0)$$

similarly

$$x(t)e^{-j\omega_0 t} \xleftrightarrow{FT} X(\omega+\omega_0)$$

⑧ Differentiation in frequency domain

If $x(t) \xleftrightarrow{FT} X(\omega)$

then $-jt\,x(t) \xleftrightarrow{FT} \frac{dX(\omega)}{d\omega}$

proof:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$\frac{dX(\omega)}{d\omega} = \frac{d}{d\omega}\left[\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt\right]$$

$$= \int_{-\infty}^{\infty} x(t)\frac{d}{d\omega}e^{-j\omega t}dt]$$

$$= \int_{-\infty}^{\infty} [x(t)\, e^{-j\omega t}\, -jt\, dt]$$

$$\frac{dX(\omega)}{d\omega} = \int_{-\infty}^{\infty} -jt\, x(t)e^{-j\omega t}dt$$

$$\frac{dX(\omega)}{d\omega} = FT[-jt\, x(t)]$$

$$\boxed{-jt\, x(t) \xleftrightarrow{FT} \frac{dX(\omega)}{d\omega}}$$

⑨ Find the FT of the signal $x(t) = te^{-at}u(t)$

$$y(t) = e^{-at}u(t) \xleftrightarrow{FT} Y(\omega) = \frac{1}{a+j\omega}$$

Apply diff in frequency domain

$$-jt\,y(t) \xleftrightarrow{FT} \frac{dY(\omega)}{d\omega}$$

$$-jte^{-at}u(t) \xleftrightarrow{FT} \frac{d}{d\omega}\left(\frac{1}{a+j\omega}\right)$$

$$-jte^{-at}u(t) \xleftrightarrow{FT} -j/(a+j\omega)^2$$

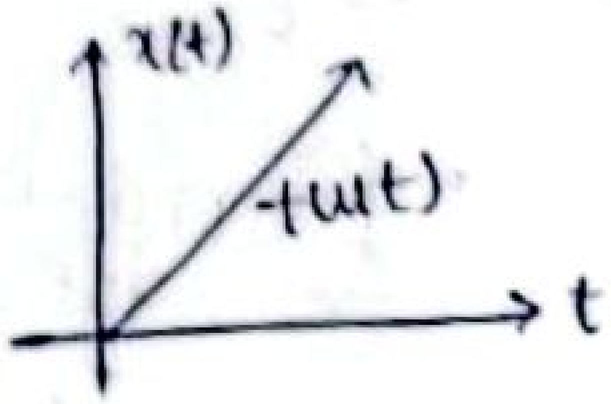
$-jt e^{-at}u(t) \xleftrightarrow{FT} \frac{-j}{(a+j\omega)^2}$

Amplitude scaling by $\frac{1}{-j}$

$$\boxed{t e^{-at}u(t) \xleftrightarrow{FT} \frac{1}{(a+j\omega)^2}}$$

If $a=0$

$$\boxed{t u(t) \xleftrightarrow{FT} \frac{-1}{\omega^2}}$$ This is called Ramp signal



(a) Modulation theorem:-

If $x(t) \xleftrightarrow{FT} X(\omega)$

then $x(t)\cos(\omega_0 t) \xleftrightarrow{FT} \frac{1}{2}[X(\omega-\omega_0)+X(\omega+\omega_0)]$

Pf:-

$FT[x(t)\cos(\omega_0 t)] = FT\left\{x(t)\left[\frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2}\right]\right\}$

$= FT\left\{\frac{1}{2}x(t)e^{j\omega_0 t}+\frac{1}{2}x(t)e^{-j\omega_0 t}\right\}$

$= \frac{1}{2}FT[x(t)e^{j\omega_0 t}]+\frac{1}{2}FT[x(t)e^{-j\omega_0 t}]$

$\because x(t)e^{j\omega_0 t} \xleftrightarrow{FT} X(\omega-\omega_0)$

$\because x(t)e^{-j\omega_0 t} \xleftrightarrow{FT} X(\omega+\omega_0)$

$\therefore FT[x(t)\cos(\omega_0 t)] = \frac{1}{2}[X(\omega-\omega_0)+X(\omega+\omega_0)]$

$x(t)\cos(\omega_0 t) \xleftrightarrow{FT} \frac{1}{2}[X(\omega-\omega_0)+X(\omega+\omega_0)]$

simillarly

$x(t)\sin(\omega_0 t) \xleftrightarrow{FT} \frac{1}{2j}[X(\omega-\omega_0)-X(\omega+\omega_0)]$

(b) Find the FT of signal $x(t)=\cos(\omega_0 t)$

$x(t)=(1)\cos(\omega_0 t)$

let $y(t)=1 \xleftrightarrow{FT} Y(\omega)=2\pi\delta(\omega)$

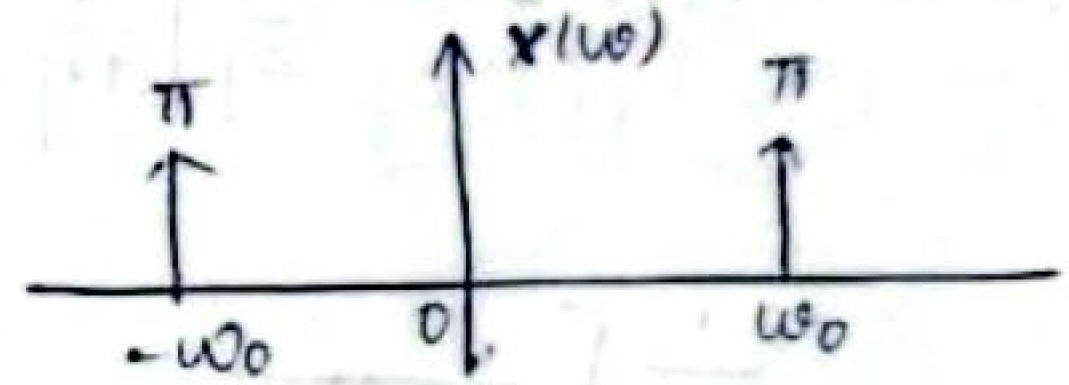
Apply modulation theorem

$y(t)\cos(\omega_0 t) \xleftrightarrow{FT} \frac{1}{2}[Y(\omega-\omega_0)+Y(\omega+\omega_0)]$

$(1)\cos(\omega_0 t) \xleftrightarrow{FT} \frac{1}{2}[2\pi\delta(\omega-\omega_0)+2\pi\delta(\omega+\omega_0)]$

$$\boxed{\cos(\omega_0 t) \xleftrightarrow{FT} \pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]}$$

This is called as pair of impulse



simillarly:

$$\boxed{\sin(\omega_0 t) \xleftrightarrow{FT} \frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]}$$

(c) $x(t) = e^{at}\cos(\omega_0 t)u(t)$

$y(t)=e^{-at}u(t) \xleftrightarrow{FT} Y(\omega)=\frac{1}{a+j\omega}$

Modulation Theorem:

$y(t)\cos(\omega_0 t) \xleftrightarrow{FT} \frac{1}{2}[Y(\omega-\omega_0)+Y(\omega+\omega_0)]$

$e^{-at}\cos(\omega_0 t)u(t) \xleftrightarrow{FT} \frac{1}{2}\left[\frac{1}{a+j(\omega-\omega_0)}+\frac{1}{a+j(\omega+\omega_0)}\right]$

$$e^{-at}\cos(\omega_0 t)u(t) \xleftrightarrow{FT} \frac{1}{2}\left[\frac{1}{(a+j\omega)-j\omega_0}+\frac{1}{(a+j\omega_0)+j\omega_0}\right]$$

$$\frac{1}{2}\left[\frac{(a+j\omega)+\cancel{j\omega_0}+(a+j\omega)-\cancel{j\omega_0}}{(a+j\omega)^2+\omega_0^2}\right]$$

$$\boxed{\therefore\ e^{-at}\cos(\omega_0 t)u(t) \xleftrightarrow{FT} \frac{(a+j\omega)}{(a+j\omega)^2+\omega_0^2}}$$

⑥ $x(t) = e^{-at}\sin(\omega_0 t)u(t)$

$$y(t) = e^{-at}u(t) \xleftrightarrow{FT} y(\omega) = \frac{1}{a+j\omega}$$

M.T

$$y(t)\sin(\omega_0 t) \xleftrightarrow{FT} \frac{1}{2j}\left[Y(\omega-\omega_0) - Y(\omega+\omega_0)\right]$$

$$e^{-at}\sin(\omega_0 t)u(t) \xleftrightarrow{FT} \frac{1}{2j}\left[\frac{1}{a+j(\omega-\omega_0)} - \frac{1}{a+j(\omega+\omega_0)}\right]$$

$$= \frac{1}{2j}\left[\frac{1}{(a+j\omega)-j\omega_0} - \frac{1}{(a+j\omega_0)+j\omega_0}\right]$$

$$= \frac{1}{2j}\left[\frac{(a+j\omega)+j\omega_0-(a+j\omega)+j\omega_0}{(a+j\omega)^2+\omega_0^2}\right]$$

$$= \frac{1}{2j}\left[\frac{2j\omega_0}{(a+j\omega)^2+\omega_0^2}\right]$$

$$e^{-at}\sin(\omega_0 t)u(t) \xleftrightarrow{FT} \frac{\omega_0}{(a+j\omega_0)^2+\omega_0^2}$$

⑩ convolution ...

If $x_1(t) \xleftrightarrow{FT} X_1(\omega)$

$x_2(t) \xleftrightarrow{FT} X_2(\omega)$

Then

$x_1(t) * x_2(t) \xleftrightarrow{FT} X_1(\omega)X_2(\omega)$

proof:

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)\,d\tau \quad ①$$

$$FT[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty}[x_1(t) * x_2(t)]e^{-j\omega t}dt$$

from ①

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau\right]e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} x(\tau)\underbrace{\left[\int_{-\infty}^{\infty} x_2(t-\tau)e^{-j\omega t}dt\right]}_{FT[x_2(t-\tau)]}d\tau$$

$$= X_2(\omega)e^{-j\omega\tau}$$

$$= \int_{-\infty}^{\infty} x_1(\tau)X_2(\omega)e^{-j\omega\tau}d\tau$$

$$= X_2(\omega)\int_{-\infty}^{\infty} x_1(\tau)e^{-j\omega\tau}d\tau$$

$$= X_2(\omega)\cdot X_1(\omega)$$

$FT[x_1(t) * x_2(t)] = X_1(\omega)X_2(\omega)$

$FT[x_1(t) * x_2(t)] = FT[x_1(\omega)]\,FT[x_2(\omega)]$

$x_1(t) * x_2(t) = IFT[X_1(\omega)X_2(\omega)]$

$$\boxed{x_1(t) * x_2(t) \xleftrightarrow{FT} X_1(\omega) X_2(\omega)}$$

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau)\, d\tau$$

$$= IFT[X_1(\omega) X_2(\omega)]$$

Q) Find the convolution b/w the signals $x(t)$ and $\delta(t)$

$x(t) = x_1(t)$ and $x_2(t) = \delta(t)$

$$x_1(t) * x_2(t) = IFT[X_1(\omega) X_2(\omega)]$$

$$X_1(\omega) X_2(\omega) = X(\omega)(1) = X(\omega)$$

$$= IFT[X(\omega)]$$

$$x(t) * \delta(t) = x(t)$$

Any function convolution with impulse result same function

Q) Find the linear convolution b/w the two rectangle pluses.

$$x_1(t) = A\,rect(t/\tau) \xleftrightarrow{FT} X_1(\omega) = A\tau\, Sa\left(\frac{\omega\tau}{2}\right)$$

$$x_2(t) = A\,rect(t/\tau) \xleftrightarrow{FT} X_2(\omega) = A\tau\, Sa\left(\frac{\omega\tau}{2}\right)$$

$$X_1(\omega) X_2(\omega) = A^2\tau^2 Sa^2\left(\frac{\omega\tau}{2}\right)$$

$$x_1(t) * x_2(t) = IFT\left[A^2\tau^2 Sa^2\left(\frac{\omega\tau}{2}\right)\right]$$

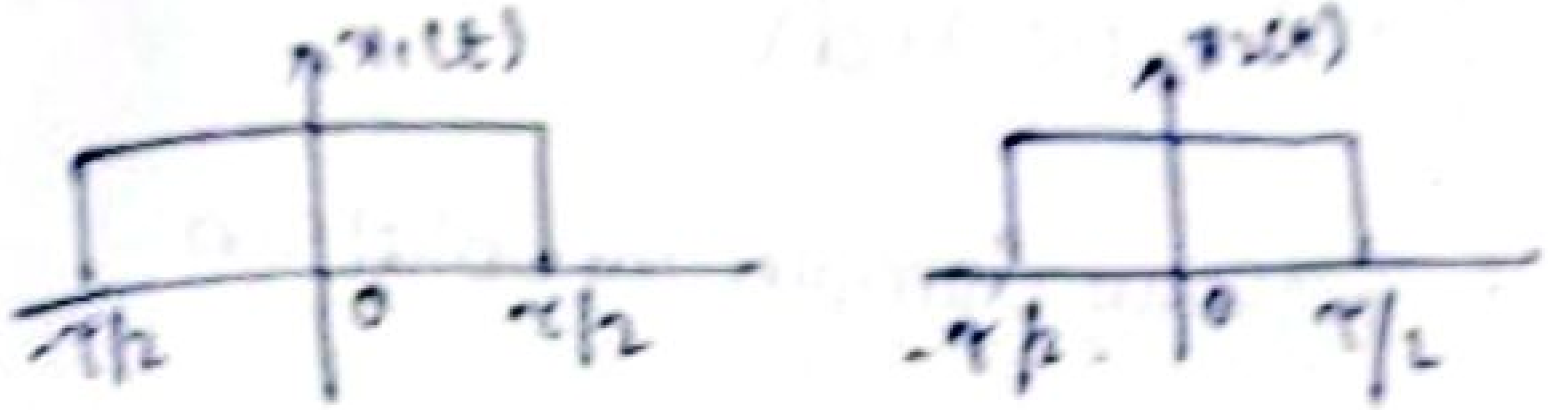
$$A\left(1 - \frac{|t|}{\tau}\right) \xleftrightarrow{FT} A\tau\, Sa^2\left(\frac{\omega\tau}{2}\right)$$

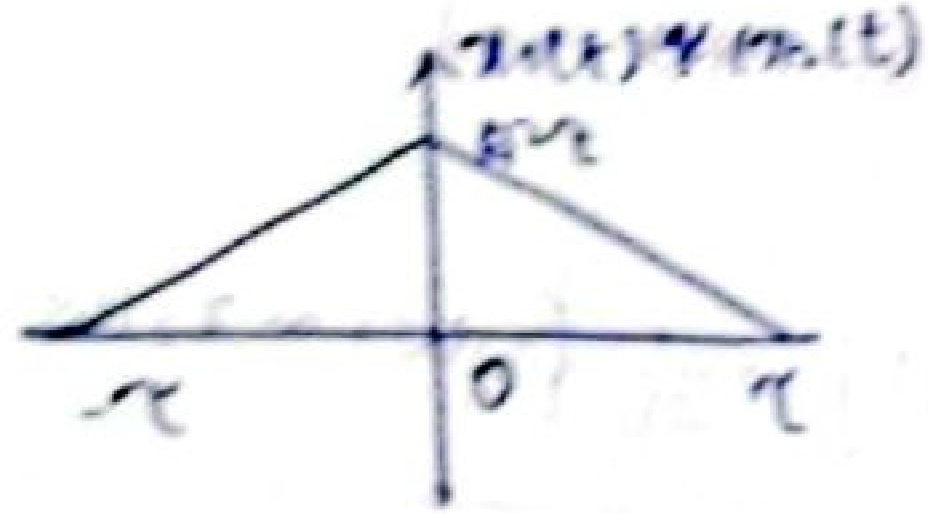
Amplitude scaling by factor $A\tau$

$$A^2\tau\left(1 - \frac{|t|}{\tau}\right) \xleftrightarrow{FT} A^2\tau^2 Sa^2\left(\frac{\omega\tau}{2}\right)$$

$$IFT\left[A^2\tau^2 Sa^2\left(\frac{\omega\tau}{2}\right)\right] = A^2\tau\left(1 - \frac{|t|}{\tau}\right)$$

$$\boxed{x_1(t) * x_2(t) = A^2\tau\left(1 - \frac{|t|}{\tau}\right)}$$





when we performing linear convolution b/w two rectangle pluses we get triangle plus.

(11) **Multiplication theorem:**

If $x_1(t) \xleftrightarrow{FT} X_1(\omega)$

$x_2(t) \xleftrightarrow{FT} X_2(\omega)$

then

$$x_1(t) x_2(t) \xleftrightarrow{FT} \frac{1}{2\pi}[X_1(\omega) * X_2(\omega)]$$

proof:

$$FT[x_1(t) x_2(t)] = \int_{-\infty}^{\infty} x_1(t) x_2(t) e^{-j\omega t}\, dt \quad -(1)$$

$$x_1(t) = IFT[X_1(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\lambda) e^{j\lambda t}\, d\lambda$$

$\lambda \to$ inverse lamida variable

$$\text{from (1)} \Rightarrow FT[x_1(t) x_2(t)] = \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\lambda) e^{j\lambda t}\, d\lambda\right] x_2(t) e^{-j\omega t}\, dt$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\lambda)\underbrace{\left[\int_{-\infty}^{\infty} x_2(t)e^{-j(\omega-\lambda)t}\,dt\right]}_{X_2(\omega-\lambda)}d\lambda$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\lambda)\,X_2(\omega-\lambda)\,d\lambda$$

This is frequency domain convolution

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega) * X_2(\omega)$$

$$\therefore FT[x_1(t)x_2(t)] = \frac{1}{2\pi}[X_1(\omega) * X_2(\omega)]$$

$$\boxed{x_1(t)x_2(t) \xleftrightarrow{FT} \frac{1}{2\pi}[X_1(\omega) * X_2(\omega)]}$$

⇒ Parseval's Theorem:-

If $x(t) \xrightarrow{FT} X(\omega)$

Then $\int_{-\infty}^{\infty}|x(t)|^2dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2 d\omega \rightarrow$ Energy

proof:

$$\int_{-\infty}^{\infty}|x(t)|^2dt = \int_{-\infty}^{\infty}x(t)x^*(t)dt \quad -①$$

$$x(t) = IFT[X(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega)e^{j\omega t}d\omega$$

$$x^*(t) = \left[\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega)e^{j\omega t}d\omega\right]^*$$

$$x^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}X^*(\omega)e^{-j\omega t}d\omega$$

from ①

$$\int_{-\infty}^{\infty}|x(t)|^2dt = \int_{-\infty}^{\infty}x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}X^*(\omega)e^{-j\omega t}d\omega\right]dt$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}X^*(\omega)\left[\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt\right]d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}X^*(\omega)X(\omega)\,d\omega$$

$$\boxed{\int_{-\infty}^{\infty}|x(t)|^2dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2d\omega} = \text{Energy}$$

Q) Find the Energy of the signal $x(t) = e^{-at}u(t)$ using parsevals theorm

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2 d\omega$$

$$x(t) = e^{-at}u(t) \xleftrightarrow{FT} X(\omega) = \frac{1}{a+j\omega}$$

$$|X(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$$

$$|X(\omega)|^2 = \frac{1}{a^2+\omega^2}$$

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{a^2+\omega^2}d\omega$$

As $a^2+\omega^2$ is even function

$$\therefore E = \frac{1}{\cancel{2}\pi}(\cancel{2})\int_{0}^{\infty}\frac{1}{a^2+\omega^2}d\omega$$

$$=\frac{1}{\pi a}\left[\tan^{-1}\left(\frac{\omega}{a}\right)\right]_0^{\infty}$$

$$=\frac{1}{\pi a}[\tan^{-1}(\infty) - \tan^{-1}0] \Rightarrow \frac{1}{\pi a}\left[\frac{\pi}{2}-0\right] = \frac{1}{2a}$$

(13) Area under the time domain signal

If $x(t) \xleftrightarrow{FT} X(\omega)$

then $\int_{-\infty}^{\infty} x(t)\,dt = X(0)$

proof:-

$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$

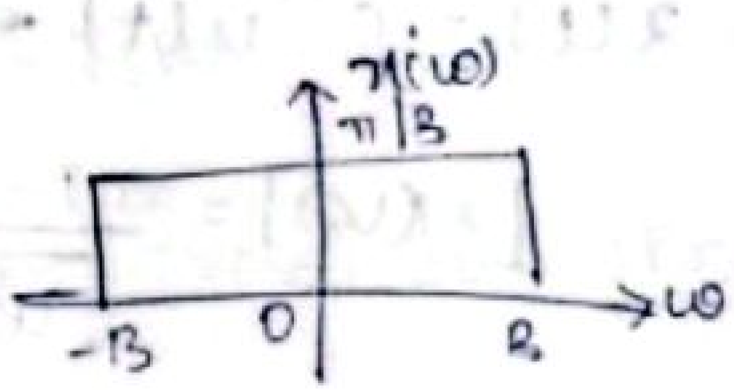
put $\omega = 0$

$$\boxed{\therefore X(0) = \int_{-\infty}^{\infty} x(t)\,dt}$$

(14) Evaluate the integral $\int_{-\infty}^{\infty} sa(Bt)\,dt$

$x(t) = sa(Bt) \xleftrightarrow{FT} \frac{\pi}{B} rect\left(\frac{\omega}{2B}\right) = X(\omega)$

$\int_{-\infty}^{\infty} x(t)\,dt = X(0)$



$\int_{-\infty}^{\infty} sa(Bt)\,dt = \frac{\pi}{B}$

similarly $\int_{-\infty}^{\infty} sa(t)\,dt = \pi$

and $\int_{-\infty}^{\infty} sa(\pi t)\,dt = \int_{-\infty}^{\infty} sinc(t)\,dt = 1$

(15) Evaluate the integral $\int_{-\infty}^{\infty} sa^2(Bt)\,dt$

$x(t) = sa^2(Bt) \xleftrightarrow{FT} \frac{\pi}{B}\left(1 - \frac{|\omega|}{2B}\right) = X(\omega)$

$\therefore \int_{-\infty}^{\infty} x(t)\,dt = X(0)$

$\int_{-\infty}^{\infty} sa^2(Bt)\,dt = \frac{\pi}{B}$

similarly $\int_{-\infty}^{\infty} sa^2(t)\,dt = \pi$

$\int_{-\infty}^{\infty} sa^2(\pi t)\,dt = \int_{-\infty}^{\infty} sinc^2(t)\,dt = 1$

(16) Area under the frequency domain signal

If $x(t) \xleftrightarrow{FT} X(\omega)$

then $\int_{-\infty}^{\infty} X(\omega)\,d\omega = 2\pi x(0)$

proof:-

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, dt$

put $t = 0$

$$\boxed{2\pi x(0) = \int_{-\infty}^{\infty} X(\omega)\,d\omega}$$

(17) Evaluate the integral $\int_{-\infty}^{\infty} \frac{1}{1+\omega^2}\,d\omega$

$e^{-a|t|} \xleftrightarrow{FT} \frac{2a}{a^2+\omega^2}$

$e^{-|t|} \xleftrightarrow{FT} \frac{2}{1+\omega^2}$

$x(t) = \frac{1}{2} e^{-|t|} \xleftrightarrow{FT} \frac{1}{1+\omega^2} = X(\omega)$

$\int_{-\infty}^{\infty} X(\omega)\,d\omega = 2\pi x(0)$

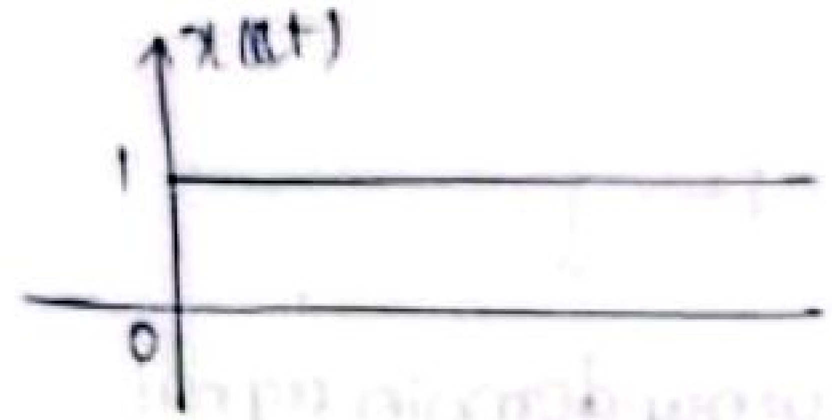
$x(t) = \frac{1}{2} e^{-|t|}$

$x(0) = \frac{1}{2}$

$= 2\pi\left(\frac{1}{2}\right)$

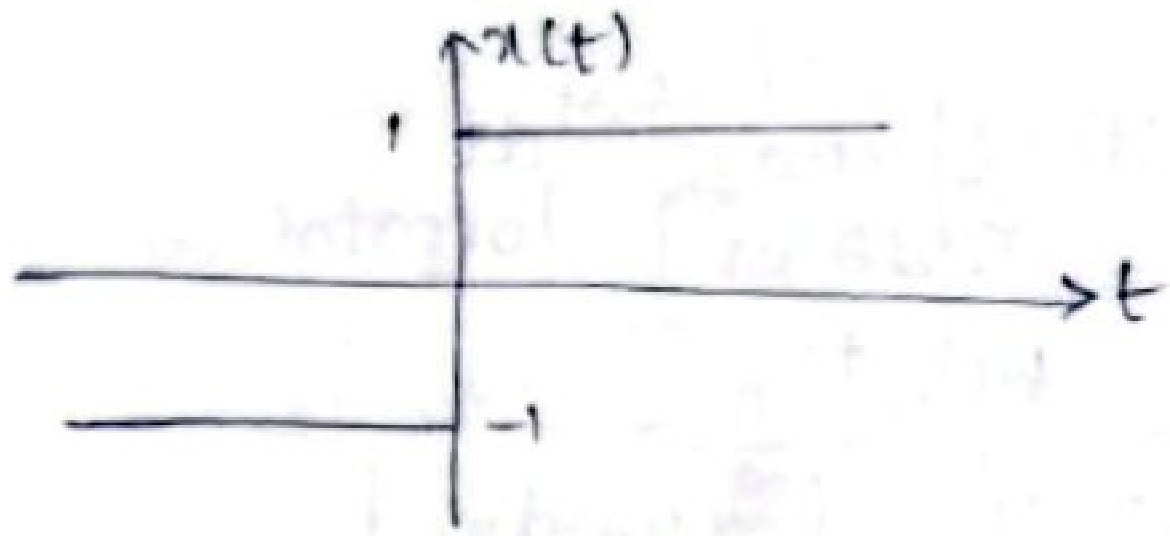
$= \pi$

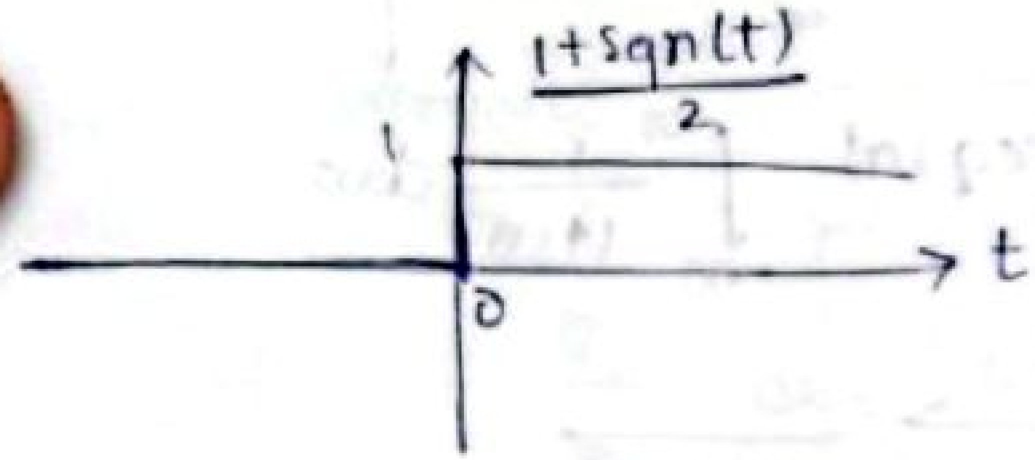
$x(t) = u(t)$



$$\int_{-\infty}^{\infty} |x(t)|\,dt = \int_{0}^{\infty} 1\, dt = \infty$$

Not absolutely Integrable.





$u(t) = \frac{1}{2} + \frac{1}{2}\, sgn(t)$

$1 \xleftrightarrow{FT} 2\pi\delta(\omega)$

Amplitude scaling operation

$\frac{1}{2} \xleftrightarrow{FT} \frac{2\pi}{2}\delta(\omega)$

$\frac{1}{2} \xleftrightarrow{FT} \pi\delta(\omega)$

$sgn(t) \xleftrightarrow{FT} \frac{2}{j\omega}$

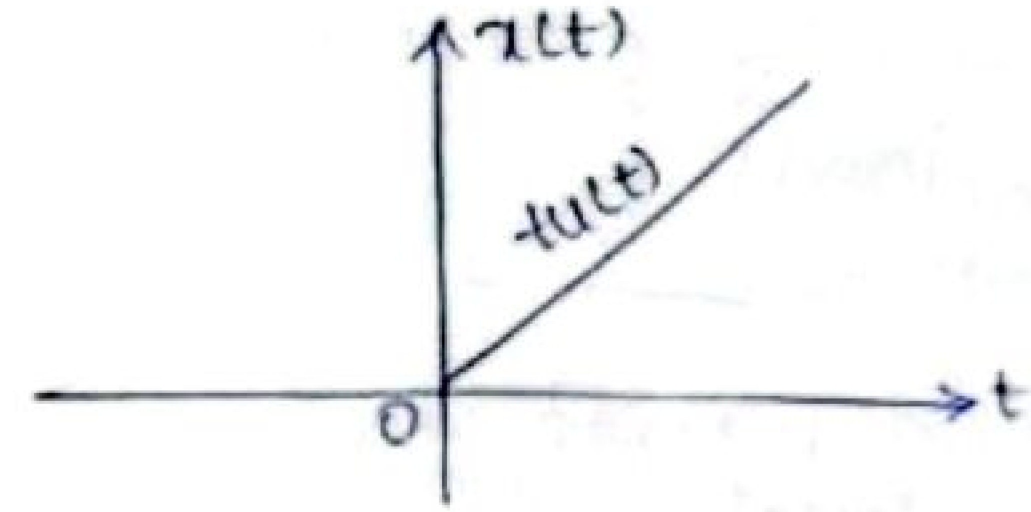
$\frac{1}{2}\, sgn(t) \xleftrightarrow{FT} \frac{1}{j\omega}$

$= \pi\delta(\omega) + \frac{1}{j\omega}$

$$\boxed{u(t) \xleftrightarrow{FT} \frac{1}{j\omega} + \pi\delta(\omega)}$$

⑤ Find the FT of the Ramp signal

$x(t) = t\,u(t)$



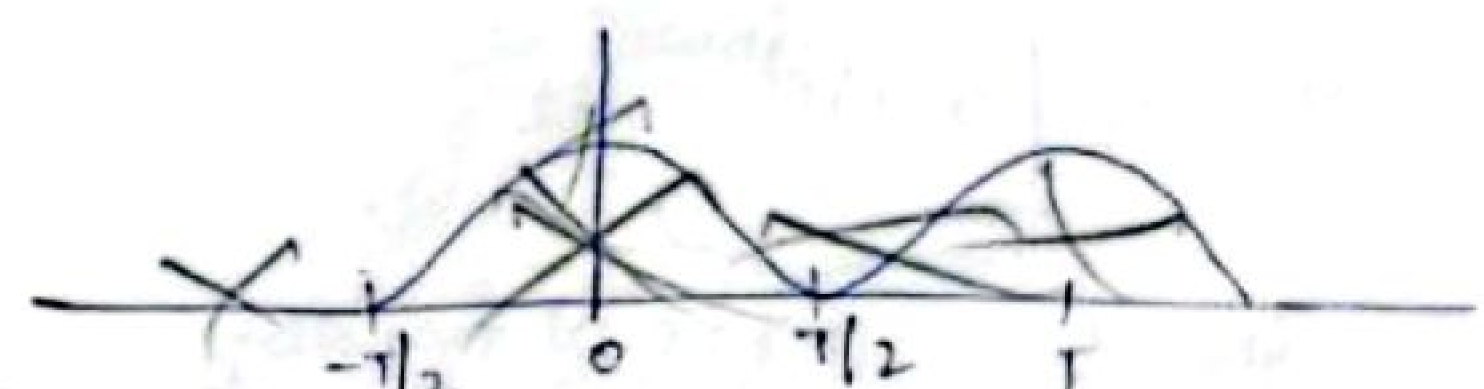
$y(t) = t e^{-at} u(t) \xleftrightarrow{FT} Y(\omega) = \frac{1}{(a+j\omega)^2}$

put $a = 0$

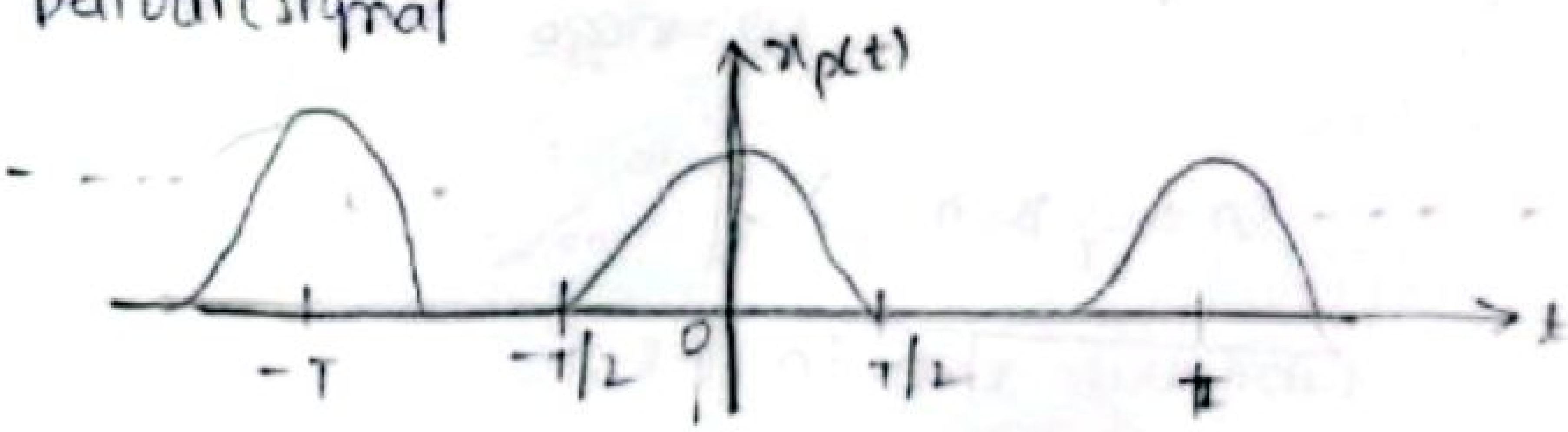
$t\,u(t) \xleftrightarrow{FT} \frac{1}{(j\omega)^2}$.

$$\boxed{t\,u(t) \xleftrightarrow{FT} \frac{-1}{\omega^2}}$$

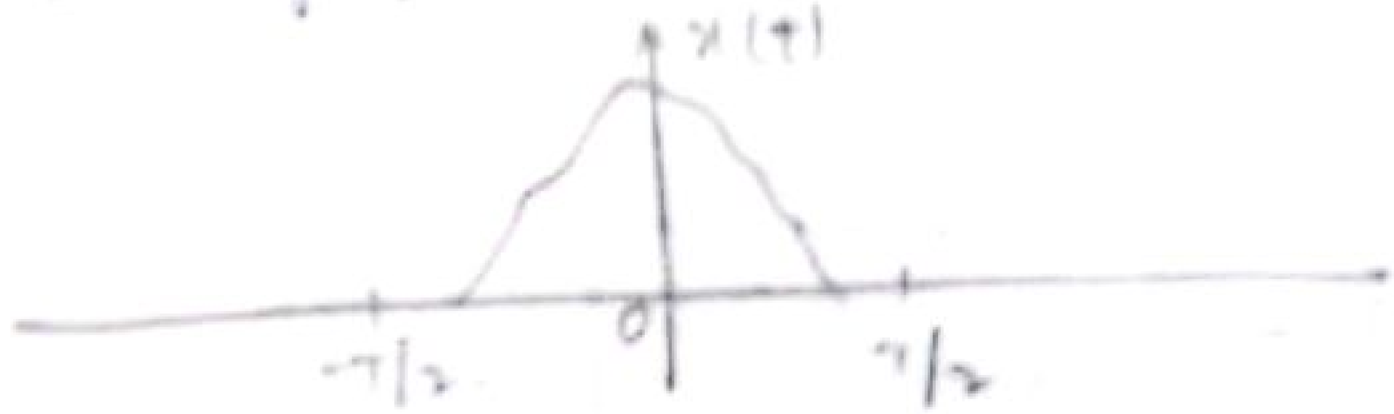
Fourier Transform of the periodic signal



periodic signal



Aperiodic signal



$$x(t) = \begin{cases} x_p(t) & \text{for } -T/2 \le t \le T/2 \\ 0 & \text{otherwise} \end{cases}$$

EFS of $x_p(t)$

$$x_p(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$C_n = \frac{1}{T}\int_{t_0}^{t_0+T} x_p(t) e^{-jn\omega_0 t}\, dt$$

Let $t_0 = -T/2$

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} x_p(t) e^{-jn\omega_0 t}\, dt$$

$x_p(t) \longleftrightarrow x(t)$

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jn\omega_0 t}\, dt$$

$$C_n = \frac{1}{T}\underbrace{\int_{-\infty}^{\infty} x(t) e^{-jn\omega_0 t}\, dt}_{\text{FTs at frequency } \omega = n\omega_0}$$

$C_n = \frac{1}{T}$

$$C_n = \frac{1}{T} X(n\omega_0)$$

substitude this in EFS

$$x_p(t) = \sum_{n=-\infty} \frac{1}{T} X(n\omega_0) e^{jn\omega_0 t}$$

$$\boxed{x_p(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(n\omega_0) e^{jn\omega_0 t}}$$

Apply FT on Both sides

$$FT[x_p(t)] = FT\left[\frac{1}{T}\sum_{n=-\infty}^{\infty} X(n\omega_0) e^{jn\omega_0 t}\right]$$

$$FT[x_p(t)] = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(n\omega_0)\, FT[e^{jn\omega_0 t}]$$

$$1 \xleftrightarrow{FT} 2\pi\delta(\omega)$$

Apply frequency shifting

$$1 \cdot e^{jn\omega_0 t} \xleftrightarrow{FT} 2\pi\delta(\omega - n\omega_0)$$

$$FT[e^{jn\omega_0 t}] = 2\pi\delta[\omega - n\omega_0]$$

$$\therefore FT[x_p(t)] = \frac{2\pi}{T}\sum_{n=-\infty} X(n\omega_0)\,\delta[\omega - n\omega_0]$$

$$\boxed{x_p(t) \xleftrightarrow{FT} \frac{2\pi}{T}\sum_{n=-\infty}^{\infty} X[n\omega_0]\,\delta(\omega - n\omega_0)}$$

	$x(t)$	
1	$\delta(t)$	1
2	$e^{-at}u(t)$	$\frac{1}{a+j\omega}$
3	$e^{at}u(t)$	$\frac{1}{a-j\omega}$
4	$A\Pi(t/\tau)$	$A\tau\, Sa\left(\frac{\omega\tau}{2}\right)$
5	$A\left(1-\frac{\lvert t\rvert}{\tau}\right)$	$A\tau\, Sa^2\left(\frac{\omega\tau}{2}\right)$
6	$e^{-a\lvert t\rvert}$	$\frac{2a}{a^2+\omega^2}$
7	$sgn(t)$	$\frac{2}{j\omega}$
8	$Sa(Bt)$	$\frac{\pi}{B}\, rect\left(\frac{\omega}{2B}\right)$
9	$Sa^2(Bt)$	$\frac{\pi}{B}\left(1-\frac{\lvert\omega\rvert}{2B}\right)$
	1	$2\pi\delta(\omega)$
11	$a_1x_1(t)+a_2x_2(t)$	$a_1X_1(\omega)+a_2X_2(\omega)$
12	$x(t-t_0)$	$X(\omega)e^{-j\omega t_0}$
13	$x(t+t_0)$	$X(\omega)e^{j\omega t_0}$
14	$\delta(t-1)$	$e^{-j\omega}$
15	$\delta(t+1)$	$e^{j\omega}$
16	Duality property $x(t)$	$2\pi x(-\omega)$
17	diff in Time $\frac{dx(t)}{dt}$	$j\omega X(\omega)$
18	→ Frequency shifting $x(t)e^{j\omega_0 t}$	$X(\omega-\omega_0)$
	→ $x(t)e^{-j\omega_0 t}$	$X(\omega+\omega_0)$
19	diff in freq $-jtx(t)$	$\frac{dX(\omega)}{d\omega}$
20	Modulation → $x(t)\cos(\omega_0 t)$	$\frac{1}{2}[X(\omega-\omega_0)+X(\omega+\omega_0)]$
	→ $x(t)\sin(\omega_0 t)$	$\frac{1}{2j}[X(\omega-\omega_0)-X(\omega+\omega_0)]$
21	Convolution $x_1(t)*x_2(t)$	$X_1(\omega)X_2(\omega)$
22	Multiplication $x_1(t)x_2(t)$	$\frac{1}{2\pi}[X_1(\omega)*X_2(\omega)]$
23	Parsevals $\int_{-\infty}^{\infty}\lvert x(t)\rvert^2 dt$	$\frac{1}{2\pi}\int_{-\infty}^{\infty}\lvert X(\omega)\rvert^2 d\omega$
24	$X(0)$	$\int_{-\infty}^{\infty}x(t)dt$ A.U Time domain
25	$2\pi x(0)$	$\int_{-\infty}^{\infty}X(\omega)dt$ A.U freq domain
26	$u(t)$	$\frac{1}{j\omega}+\pi\delta(\omega)$
27	$tu(t)$	$\frac{-1}{\omega^2}$
28	$x_p(t)$	$\frac{2\pi}{T}\sum_{n=-\infty}^{\infty}X[n\omega_0]\,\delta(\omega-n\omega_0)$

EFT

① $\alpha x(t)+\beta y(t)$ — $\alpha C_n+\beta d_n$

② $x(\alpha t)$ — C_n (Time scaling)

③ $x(-t)$ — C_{-n} (Time reversal)

④ $\rightarrow x(t-t_0)$ — $C_n e^{-jn\omega_0 t_0}$ (Time shifting)

$\rightarrow x(t+t_0)$ — $C_n e^{jn\omega_0 t_0}$

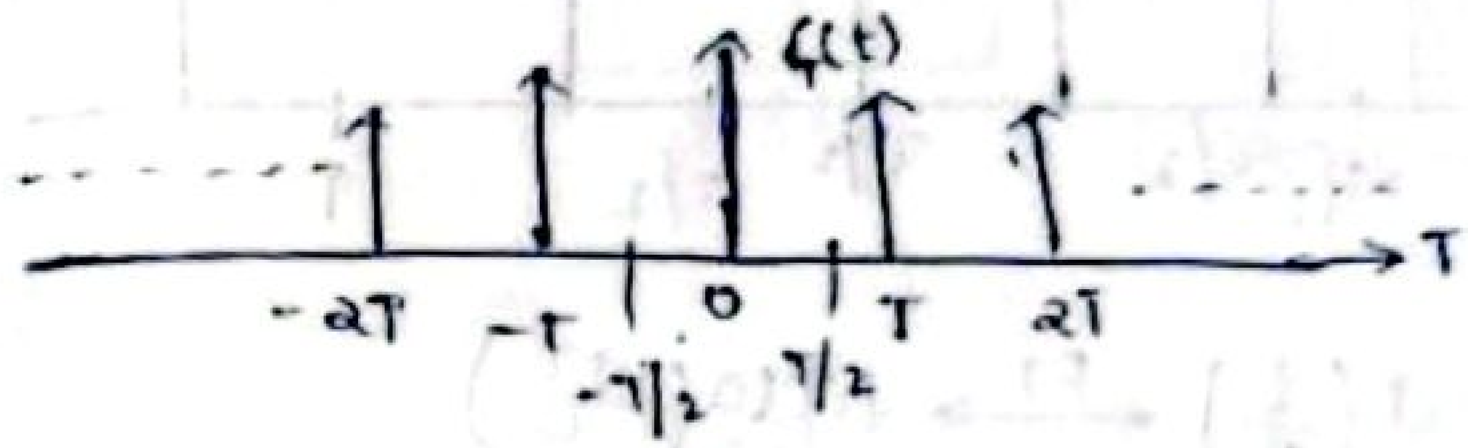
⑤ $x(t)e^{jm\omega_0 t}$ — C_{n-m} [frequency shift]

⑥ $x^*(t)$ — C_{-n}^* [conjugation]

⑤ Find the FT of train of impulses

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

$$\cdots\ \delta(t+T)+\delta(t)+\delta(t-T)+\cdots$$



$$F[\delta_T(t)] = \omega_0 \sum_{n=-\infty}^{\infty} x(n\omega_0)\,\delta(\omega-\omega_0)$$

$$x(t)=\delta(t) \xleftrightarrow{FT} x(\omega)=1$$

$$x(n\omega_0)=1$$

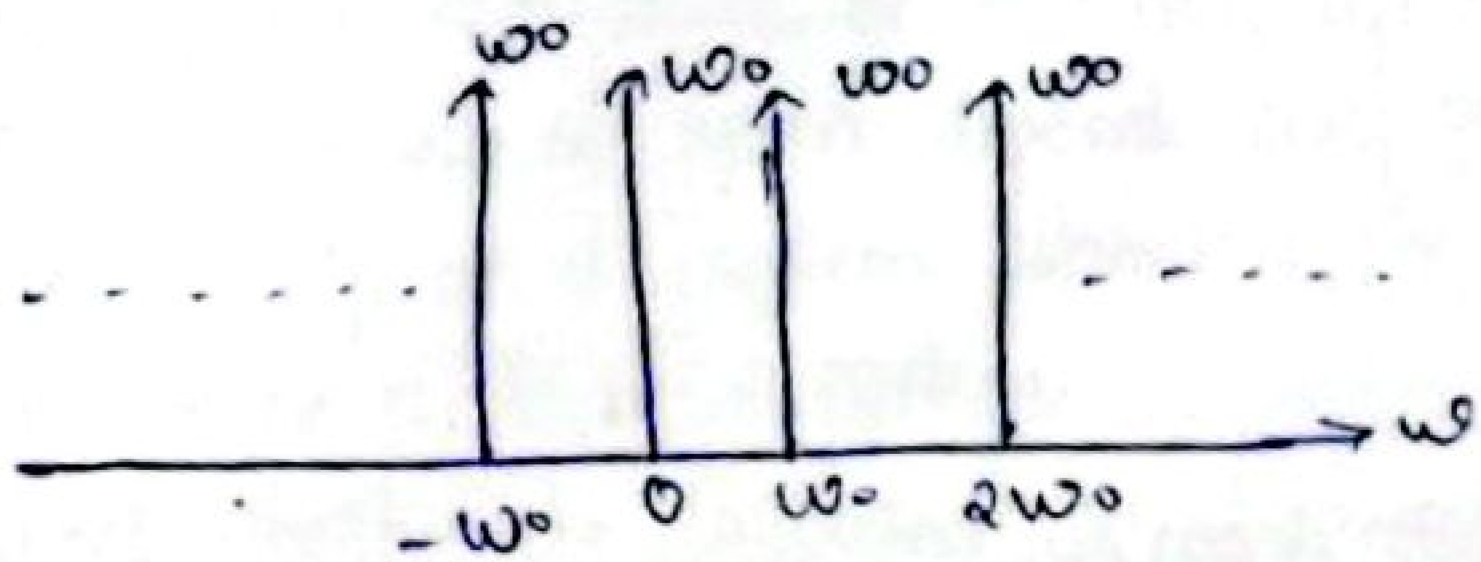
$$FT[\delta_T(t)] = \omega_0 \sum_{n=-\infty}^{\infty} 1\,\delta(\omega-n\omega_0)$$

$$\boxed{\therefore\ \delta_T(t) \xleftrightarrow{FT} \omega_0 \sum_{n=-\infty}^{\infty} \delta(\omega-n\omega_0)}$$

$$x_T(t) = FT[\delta_T(t)] = \omega_0 \sum_{n=-\infty}^{\infty} \delta(\omega-n\omega_0)$$

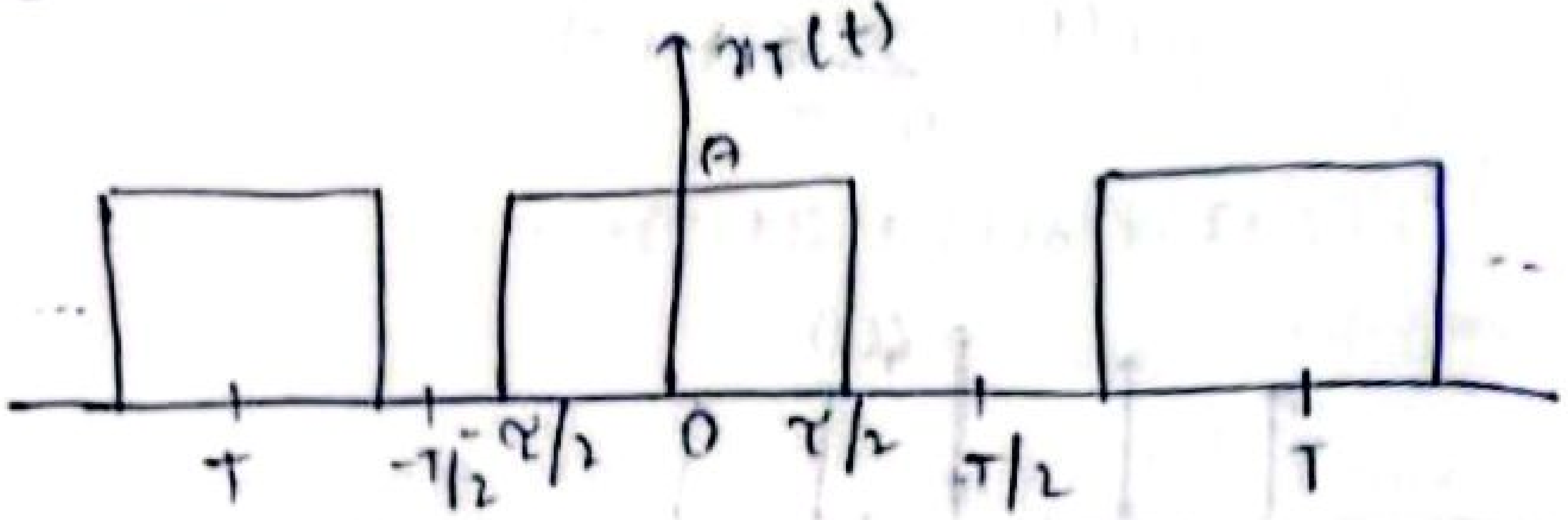
$$x_T(t) \xleftrightarrow{FT} X_T(\omega)$$

$$X_T(\omega) \longleftrightarrow \omega_0 \sum_{n=-\infty}^{\infty} \delta(\omega-n\omega_0)$$



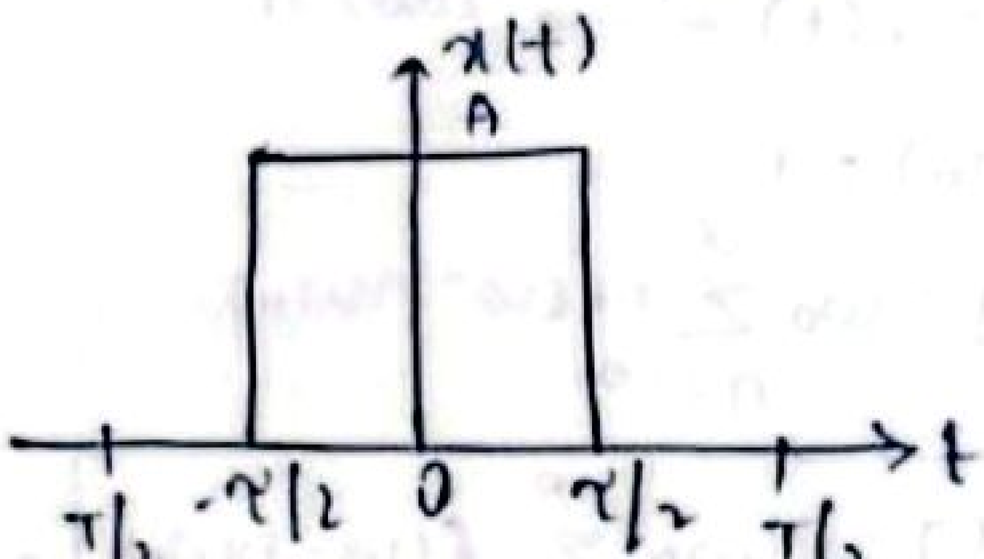
FT of train of impulses is again train of impulses

(Q) Find the FT of train of ... pulses



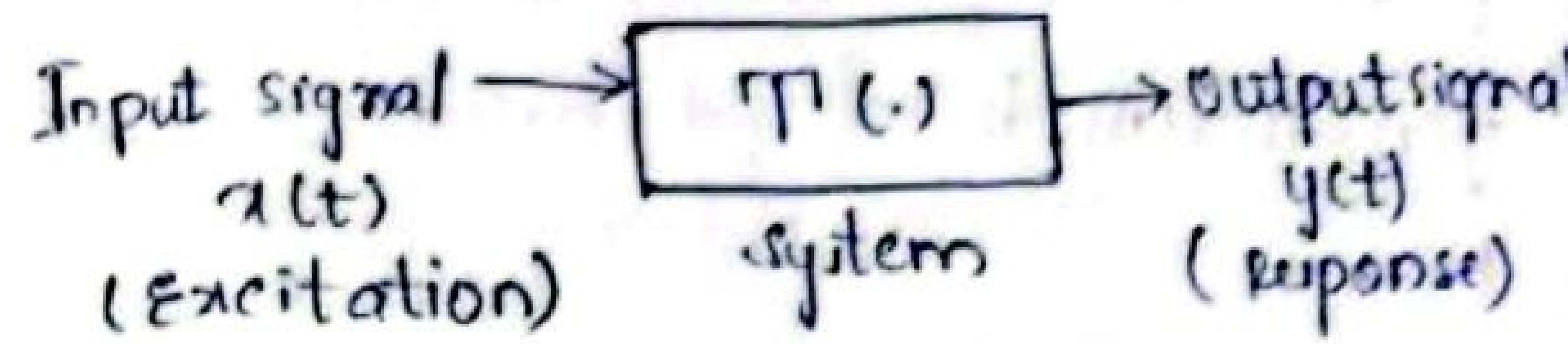
$$A\,\text{rect}\left(\frac{t}{\tau}\right) \xleftrightarrow{FT} A\tau\, sa\left(\frac{\omega\tau}{2}\right)$$

$$x_T(\omega) = FT[x_T(t)] = \omega_0 \sum_{n=-\infty}^{\infty} x(n\omega_0)\,\delta(\omega - n\omega_0)$$



$$\therefore x(n\omega_0) = A\tau\, sa\left(\frac{n\omega_0\tau}{2}\right)$$

$$\boxed{x_T(\omega) = \omega_0 \sum_{n=-\infty}^{\infty} A\tau\, sa\left(\frac{n\omega_0\tau}{2}\right)\delta(\omega - n\omega_0)}$$



System equation

$$y(t) = T[x(t)]$$

$y(t)$ ↓ output signal; $x(t)$ ↓ Input signal

$$x(t) \xrightarrow{T(\cdot)} y(t) = T[x(t)]$$

$$x(t) \longrightarrow y(t) = T[x(t)]$$

continuos time system is a device that operates on the continuos time I/p signal to produce continuos time o/p signal

Different types of system

① static vs Dynamic system:

static system is also know as memory less system. the system is said to be a static system if and only if the o/p of system depends only on the present I/p of the system. Otherwise the system is said to be a Dynamic system.

(Q) check whether the following system is static (or) Dynamic

$$y(t) = ax(t) + b$$

let $t=0$

$y(0) = a(x(0))+b$

at $t=+1$

$y(1) = a(x(1))+b$

at $t=-1$

$y(-1) = ax(-1)+b$

present o/p depents only on the present I/p ∴ the system is static system

② $y(t) = x(t+1)$

at $t=0$

$y(0) = x(1)$

present o/p depents on further I/p ∴ system is dynamic system.

③ $y(t) = x(-t)$

$t=0$

$y(0) = x(0)$

$t=+1$

$y(1) = x(-1)$

∴ present o/p depents on past I/p

∴ dynamic system

⑥ Linear vs non linear

system is said to be linear system. if and if only the system satisfies the

① Homoginity principle

② superposition principle

they ... system is linear system otherwise non-linear.

① If input is scaled by factor 'a' then o/p is also scaled by factor 'a' then system is Homoginity principle satifies

let $x(t) \xrightarrow{T(\cdot)} y(t)$

$ax(t) \longrightarrow ay(t)$

②

$x_1(t) \longrightarrow y_1(t)$

$x_2(t) \longrightarrow y_2(t)$

$x_1(t)+x_2(t) \longrightarrow y_1(t)+y_2(t)$

This is called superposition principle.

steps to check the system is linear (or) not

I/p o/p

$x(t) \xrightarrow{T(\cdot)} y(t) = T[x(t)]$

① $x_1(t) \longrightarrow y_1(t) = T[x_1(t)]$

② $x_2(t) \longrightarrow y_2(t) = T[x_2(t)]$

③ $a_1x_1(t) + a_2x_2(t) \longrightarrow y_3(t) = T[a_1x_1(t)+a_2x_2(t)]$

④ $y_4(t) = a_1y_1(t) + a_2y_2(t)$

if $y_3(t) = y_4(t)$

system is linear system.

$T[a_1x_1(t)+a_2x_2(t)] = a_1y_1(t)+a_2y_2(t)$

$T[a_1x_1(t)+a_2x_2(t)] = a_1T[x_1(t)] + a_2T[x_2(t)]$

(a) non linear

$$y(t) = \alpha x(t) + \beta$$

① $x_1(t) \longrightarrow y_1(t) = \alpha x_1(t) + \beta$

② $x_2(t) \longrightarrow y_1(t) = \alpha x_2(t) + \beta$

③ $a_1x_1(t) + a_2x_2(t) \longrightarrow y_3(t) = \alpha[a_1x_1(t) + a_2x_2(t)] + \beta$

④ $y_4(t) = a_1y_1(t) + a_2y_2(t)$

$= a_1[\alpha x_1(t) + \beta] + a_2[\alpha x_2(t) + \beta]$

$y_3(t) \neq y_4(t)$

∴ system is non linear system

⑧ $y(t) = x(t) + \dfrac{1}{x(t-1)}$

$x_1(t) \to y_1(t) = x_1(t) + \dfrac{1}{x_1(t-1)}$ - ①

$x_2(t) \to y_2(t) = x_2(t) + \dfrac{1}{x_2(t-1)}$ - ②.

$a_1x_1(t) + a_2x_2(t) \to y_3(t) = [a_1x_1(t) + a_2x_2(t)] + \dfrac{1}{a_1x_1(t-1) + a_2x_2(t-1)}$

$y_4(t) = a_1y_1(t) + a_2y_2(t) = a_1\left[x_1(t) + \dfrac{1}{x_1(t-1)}\right] + a_2\left[x_2(t) + \dfrac{1}{x_2(t-1)}\right]$

$y_3(t) \neq y_4(t)$

so, given is non-linear system.

$y(t) = e^{xt}$

(ii) $\dfrac{d^2y(t)}{dt^2} + 2\dfrac{dy(t)}{dt} + 3y(t) = x(t)$

$\dfrac{d^2y_1(t)}{dt^2} + 2\dfrac{dy_1(t)}{dt} + 3y_1(t) = x_1(t)$

$\dfrac{d^2y_2(t)}{dt^2} + \dfrac{2dy_2(t)}{dt} + 3y_2(t) = x_2(t)$

$a_1x_1(t) + a_2x_2(t) \to a_1y_1(t) + a_2y_2(t)$

$\Rightarrow \dfrac{d^2}{dt^2}[a_1y_1(t) + a_2y_2(t)] + 2\dfrac{d}{dt}[a_1y_1(t) + a_2y_2(t)] + 3[a_1y_1(t) + a_2y_2(t)] = a_1x_1(t) + a_2x_2(t)$ ③

Linear combination ① and ②

$a_1\left[\dfrac{d^2y_1(t)}{dt^2} + \dfrac{2dy_1(t)}{dt} + 3y_1(t)\right] = a_1x_1(t) + a_2x_2(t)$ - ④

③ = ④ Linear system

⑥ $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$

$x_1(t) \to y_1(t)$

$y_1(t) = \int_{-\infty}^{t} x_1(\tau)d\tau$ - ①

$x_2(t) \to y_2(t)$

$y_2(t) = \int_{-\infty}^{t} x_2(\tau)d\tau$ - ②

$a_1x_1(t) + a_2x_2(t) \to a_1y_1(t) + a_2y_2(t)$

$a_1y_1(t) + a_2y_2(t) = \int_{-\infty}^{t} [a_1x_1(\tau) + a_2x_2(\tau)]d\tau$ - ③

$a_1y_1(t) + a_2y_2(t) = a_1\int_{-\infty}^{t} x_1(\tau)d\tau + a_2\int_{-\infty}^{t} x_2(\tau)d\tau$ - ④

③ = ④ linear system

Time variant vs Time invariant system

If the i/p & o/p characteristics of the system does not change wrt time such kind of system is said to be a time invariant system

i/p o/p

$x(t) \longrightarrow y(t) = T[x(t)]$

③ Delayed i/p

$x(t-t_0) \longrightarrow y(t, t_0) = T[x(t-t_0)]$

if $y(t, t_0) = y(t-t_0)$ then system is time-invariant

if not time varying system

⑥ Time varying or invariant system

① $y(t) = ax(t) + b$

s-1) delayed i/p

$x(t) \xrightarrow[i/p]{} \boxed{T(\cdot)} \longrightarrow y(t)$ o/p

$x(t-t_0) \rightarrow y(t, t_0)$

$y(t) = ax(t) + b$

$y(t, t_0) = ax(t-t_0) + b$ — ①

s-2) delayed o/p $y(t-t_0)$

$y(t-t_0) = ax(t-t_0) + b$

$y(t, t_0) = y(t-t_0)$

Then system is time invariant

② $y(t) = x(-t)$

s-1) delayed i/p

$x(t-t_0) \rightarrow y(t, t_0)$

$y(t, t_0) \rightarrow x(t-t_0)$ — ①

s-2) delayed o/p $y(-t-t_0)$

$y(t-t_0) = ax(-(t-t_0))$

$y(-t, t_0) = y(-t+t_0)$

$y(t, t_0) \neq y(t-t_0)$

∴ Time varying system

③ $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$

i/p   o/p

$x(t) \longrightarrow y(t)$

$y(t) = \int_{-\infty}^{t} x(\tau) d\tau$

s-1) delayed i/p

$x(t-t_0) \rightarrow y(t, t_0)$

$y(t, t_0) = \int_{-\infty}^{t} x(\tau - t_0) d\tau$ — ①

s-2) delayed o/p

t replace with $(t-t_0)$

$y(t-t_0) = \int_{-\infty}^{t-t_0} x(\tau) d\tau$ — ②

put $\tau - t_0 = t_1$, $d\tau = dt_1$, $\tau = \infty, t_1 = -\infty$

$\tau = t, t_1 = t - t_0$

eqn ① $y(t, t_0) = \int_{-\infty}^{t-t_0} x(t_1) dt_1$

$t_1 \rightarrow t$

$y(t, t_0) = \int_{-\infty}^{t-t_0} x(t) dt$

$y(t, t_0) = y(t-t_0)$

Time invariant system

NOTE :- The coefficients of differential eqn are constants then the system is said to be a time-invariant system.

⑥ i/p, o/p relation is given as below, check time variant or invariant system

① $\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 3y(t) = 1\, x(t)$ constants

time invariant system

② $y(t) = \frac{dx(t)}{dt}$

time varying system

causal vs non-causal system

The system is said to be a causal system if the o/p of the system depends present i/p, past i/ps and past o/ps only causal system

(i) $y(t) = e^{x(t)}$

$t = 0$

$y(0) = e^{x(0)}$

$y(-1) = e^{x(-1)}$

$y(1) = e^{x(1)}$

o/p depends on present i/p so causal system

(ii) $y(t) = x(-t)$

$y(0) = x(0)$

$y(1) = x(-1)$

$y(-1) = x(1)$ ↓ future i/p

∴ non causal system

(iii) $y(t) = y(t-1) + x(t) + x(t-1)$

$y(0) = \underbrace{y(-1)}_{\text{past}} + \underbrace{x(0)}_{\text{present}} + \underbrace{x(-1)}_{\text{past}}$

$y(1) = \underbrace{y(0)}_{\text{past}} + \underbrace{x(1)}_{\text{present}} + \underbrace{x(0)}_{\text{past}}$

$y(-1) = y(-2) + x(-1) + x(-2)$

causal system

Convolution:

$x(t) \longrightarrow \boxed{T(\cdot)} \longrightarrow$ o/p $(y(t))$

i/p excitation; response

$y(t) = T[x(t)]$

if i/p to the system is impluse the o/p is the impluse response of the i/p

means i/p is Impluse o/p is its corresponding Impluse response of the system

when i/p to the system is an impluse signal then the corresponding reponse of the system is nothing but impluse responese of the sytem

→ the impluse ...

$\delta(t) \longrightarrow h(t)$ i.e., $h(t) = T[\delta(t)]$

impluse i/p; impluse response

step response of the system

when i/p to the system is an unit step signal the corresponding reponse of the system is called unit step reponse of the system denoted by $s(t)$

$s(t) = T[u(t)]$

$x(t) \xrightarrow{\text{i/p}} \boxed{\text{system}} \longrightarrow y(t)$ o/p

LTI system
Linear-time invariant

i/p	o/p
$x(t)$	$\longrightarrow y(t) = T[x(t)]$
$\delta(t)$	$\longrightarrow h(t) = T[\delta(t)]$
$u(t)$	$\longrightarrow s(t) = T[u(t)]$

$\delta(t-\tau) \xrightarrow{\text{TI system}} h(t-\tau) = T[\delta(t-\tau)]$ [∵ TI system]

Linearity:

$x(\tau)\delta(t-\tau) \xrightarrow[\text{system}]{\text{Homogenous}} x(\tau)h(t-\tau)$

i/p scaled with same factor; o/p also scaled with the same factor

$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \xrightarrow[\text{principle}]{\text{superposition}} \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$

linear combination of all o/ps of corresponding i/ps sampling property of an impluse is

$$\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau = x(\tau)\Big|_{at\ \tau=t} = x(t)$$

so replace τ by t

$$\underset{i/p}{x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau} \longrightarrow \underset{o/p}{y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau}$$

$$\underset{i/p}{x(t)} \longrightarrow \underset{\text{LTI system}}{\boxed{h(t)}} \longrightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

$$= x(t) * h(t)$$

the LTI system is always characterised by using its impluse response $(h(t))$

the o/p of the LTI system is always the linear convolution b/w i/p and impluse response of the system

so for any i/p signal $x(t)$ we need to know first its impluse response then o/p is represented with terms of $h(t)$

$$y(t) = x(t) * h(t)$$

linear convolution b/w two signals $x(t)$ and $h(t)$ is defined as;

$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

where $-\infty < t < \infty$.

in general also varies depending on problem but general $-\infty < t < \infty$

① commutative : $x(t) * h(t) = h(t) * x(t)$

i.e, $x(t)$ linear convolution with $h(t)$

= $h(t)$ linear convolution with $x(t)$

② Associative propeitry: $[x(t) * h_1(t)] * h_2(t)$

$$= x(t) * (h_1(t) * h_2(t))$$

③ distributive:- $x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)$

④ shifting property:- if $x(t) * h(t) = y(t)$ then,

Right shift $\rightarrow x(t-t_1) * h(t-t_2) = y(t-t_1-t_2)$

left shift $\rightarrow x(t+t_1) * h(t+t_2) = y(t+t_1+t_2)$

⑤ scaling property:- if $x(t) * h(t) = y(t)$ then

$$x(at) * h(at) = \frac{1}{|a|}\,y(at)$$

⑥ Differentative property:-

$$\frac{d}{dt}[x(t) * h(t)] = x(t) * \frac{d}{dt}h(t)$$

$$= \frac{d}{dt}(x(t)) * h(t)$$

⑦ Integrating

$$\int_{-\infty}^{t} (x(t) * h(t))\,dt = \left[\int_{-\infty}^{t} x(t)\,dt\right] * h(t)$$

$$= x(t) * \left[\int_{-\infty}^{t} h(t)\,dt\right]$$

$x(t) * h(t) \xleftrightarrow{FT} X(\omega)H(\omega)$

$x(t) * h(t) = IFT[X(\omega)H(\omega)]$

(10) Multiplication theorem

$$x(t)h(t) \xleftrightarrow{FT} \frac{1}{2\pi}[X(\omega) * H(\omega)]$$

$$X(\omega) * H(\omega) = 2\pi FT[x(t)h(t)]$$

NOTE:

A linear convolution is used to ~~covert~~ compute the output of LTI system. The output of LTI system is always the linear convolation b/w I/P and impulse respond of the system.

→ The linear convolution b/w $x(t)$ and $h(t)$ can be computed using a diff methods

(1) Time domain method (or) graphical method

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

(2) frequency domain method

$$y(t) = IFT[X(\omega)H(\omega)]$$

(11) sampling property of an implse

(1) $x(t) * \delta(t) = x(t)$ (Time domain method)

$$x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau = x(\tau)\Big|_{t=\tau} = x(t)$$

↓ sampling property

Any signal convolution with impluse signal results same signal

(2) frequency

$$x(t) * \delta(t) = IFT[X(\omega)\cdot 1] = x(t)$$

(3) find the convolution b/w $x(t) * \delta(t-t_0)$

$$x(t) * \delta(t) = x(t)$$

shifting property of convolution

If $x(t) * h(t) = y(t)$

then $x(t-t_1) * h(t-t_2) = y(t-t_1-t_2)$

$t_1 = 0, \; t_2 = t_0$

$\therefore x(t) * \delta(t-t_0) = y(t-t_0)$

(4) $x(t-t_1) * \delta(t-t_2)$

$$x(t) * \delta(t) = x(t)$$

shifting property of convolution

$$x(t-t_1) * \delta(t-t_2) = x(t-t_1-t_2)$$

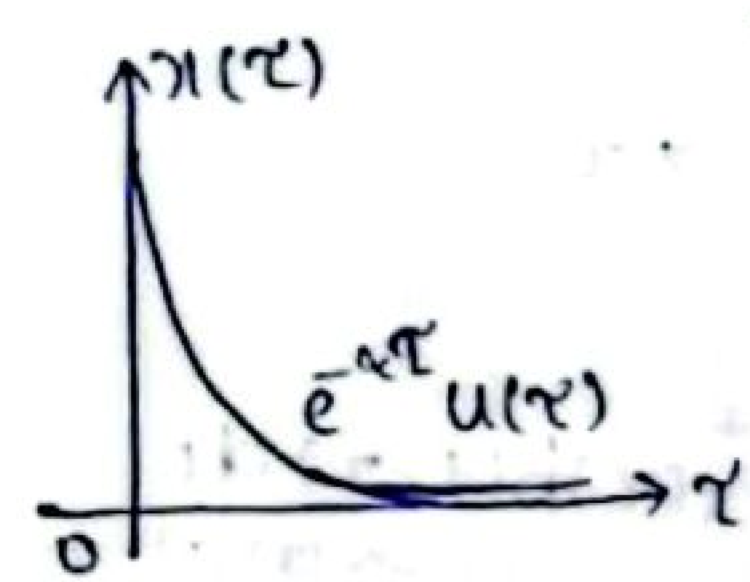
(5) find the linear convolution b/w the two following signals.

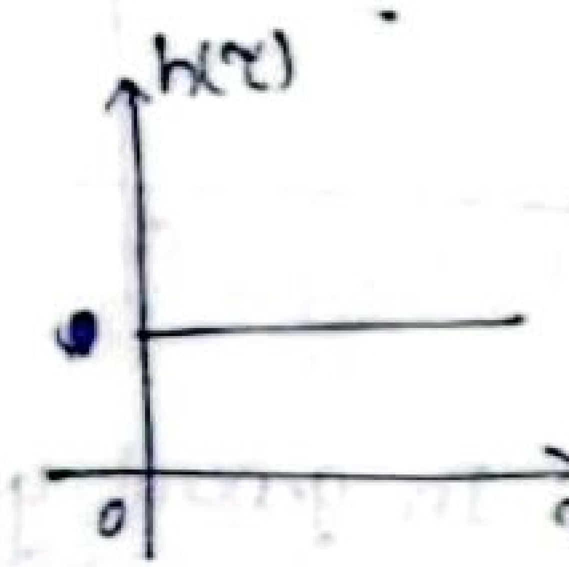
$$x(t) = e^{-\alpha t}u(t)$$

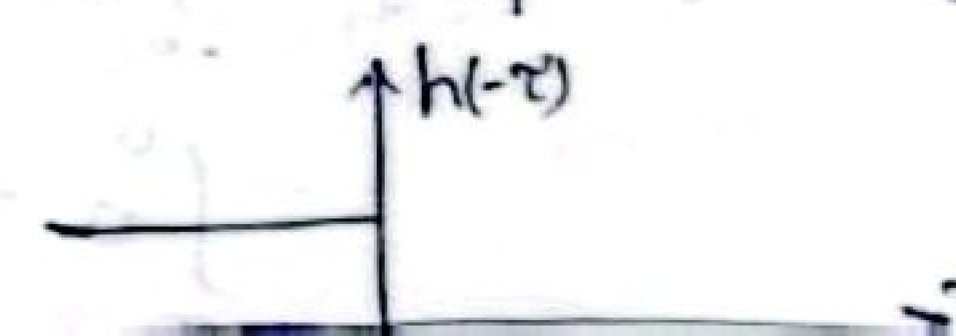
$$h(t) = u(t)$$

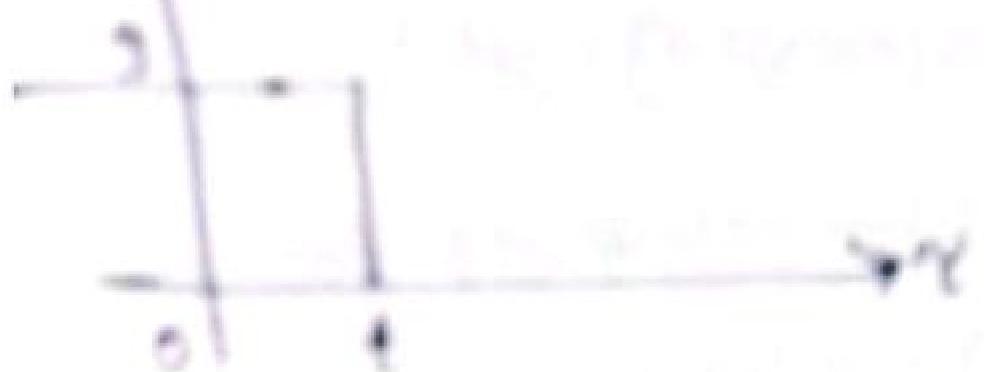
using graphical method

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$







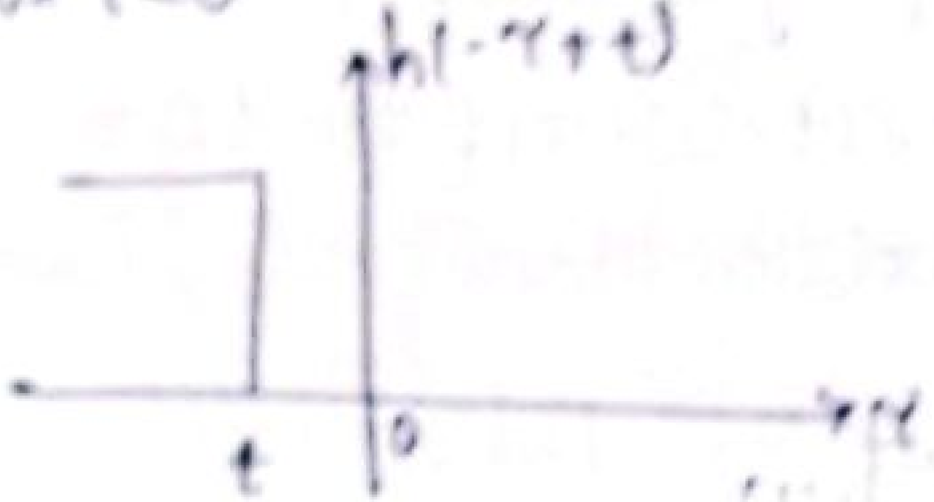


l_1 = lower limit of signal 1
u_1 = upper limit of signal 1
l_2 = lower limit of signal 2
u_2 = upper limit of signal 2

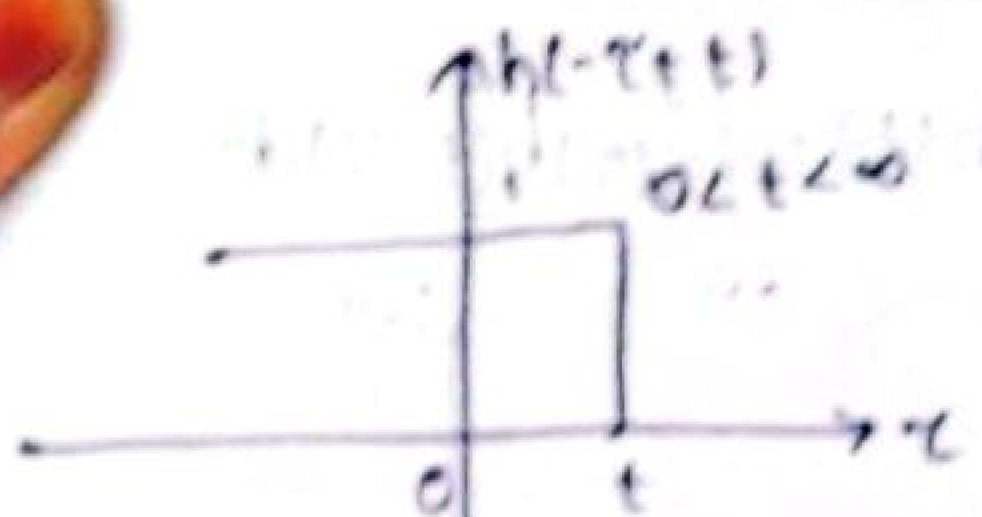
$l_1 = 0 \quad u_1 = \infty$
$l_2 = 0 \quad u_2 = \infty$

l_1+l_2	l_1+u_2	l_2+u_1	u_1+u_2
0	∞	∞	∞

for $t<0$



$\therefore y(t)=0$



In general $y(t)=\int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$

$= \int_{-\infty}^{0} 0 + \int_{0}^{t} e^{-\alpha\tau} \cdot 1 + \int_{t}^{\infty} 0$

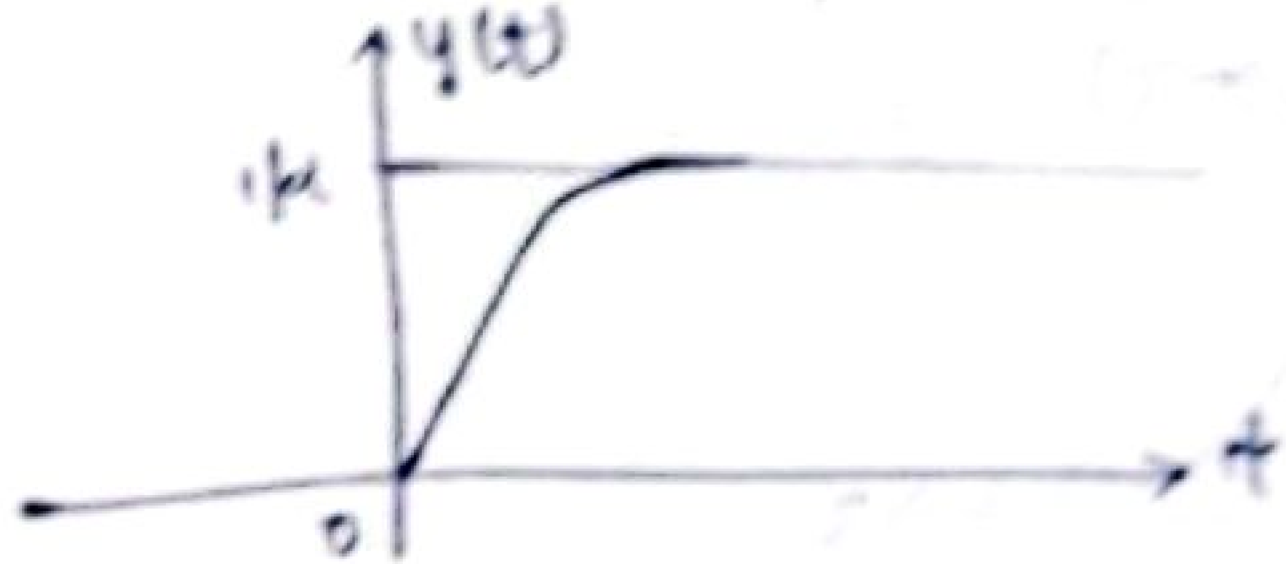
$= \left[\frac{e^{-\alpha\tau}}{-\alpha}\right]_0^t$

$= \frac{1}{-\alpha}\left[e^{-\alpha t} - e^{0}\right]$

$y(t) = \frac{1}{\alpha}\left[1-e^{-\alpha t}\right]$ for $0<t<\infty$

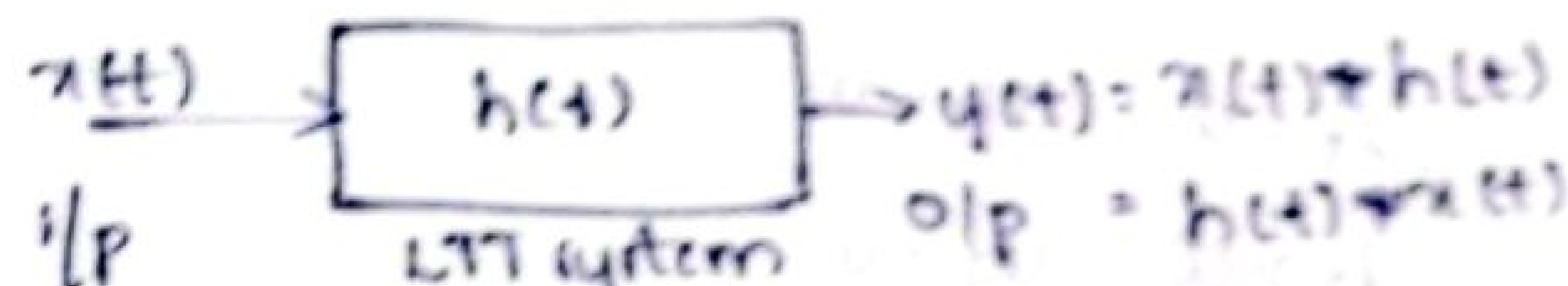
$y(t) = 0$ otherwise

$\therefore y(t) = \left(\frac{1-e^{-\alpha t}}{\alpha}\right)u(t)$



(c) The input to the LTI system is $e^{-\alpha t}u(t)$ and the impulse response of the system is $e^{\alpha t}u(-t)$ find the output of the system.

$x(t) = e^{-\alpha t}u(t)$
$h(t) = e^{\alpha t}u(-t)$



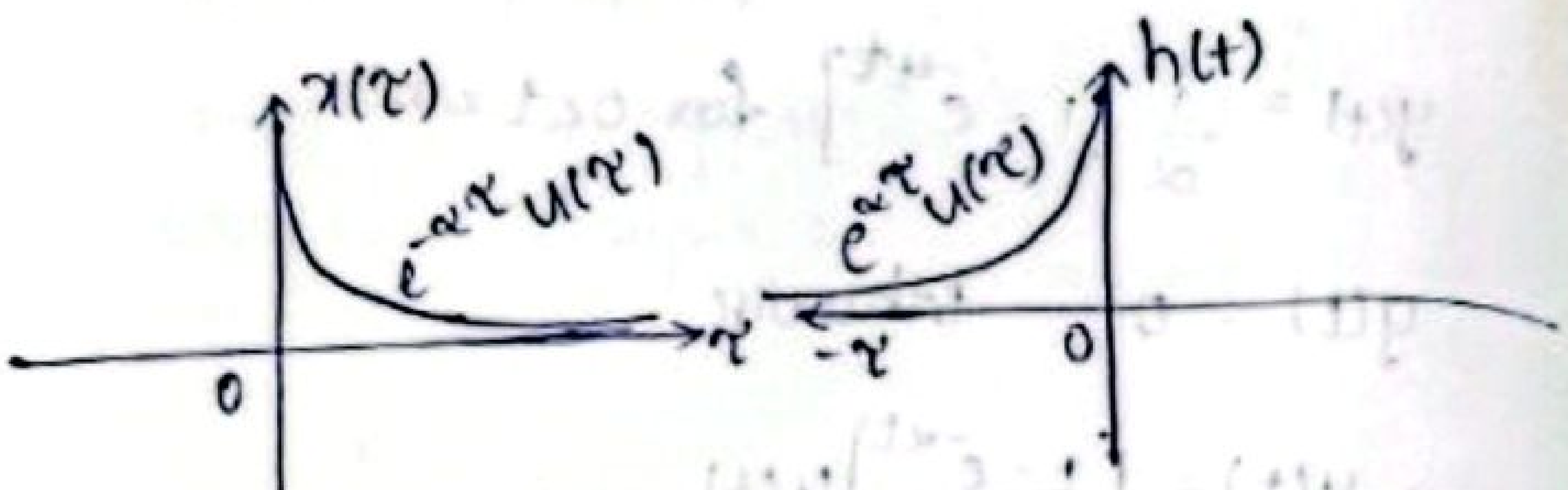
$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$

for $-\infty$ to ∞

$$y(t) = h(t) * x(t) = \int_{-\infty} h(\tau)\, x(t-\tau)\, d\tau$$

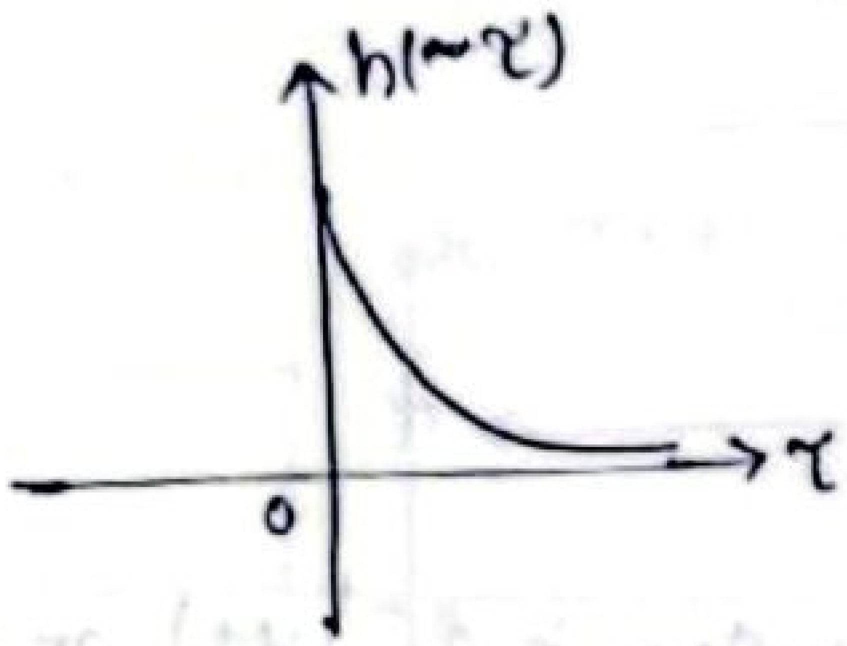
$\because$ $x(\tau) = e^{-\alpha\tau} u(\tau)$

$h(\tau) = e^{\alpha\tau} u(-\tau)$

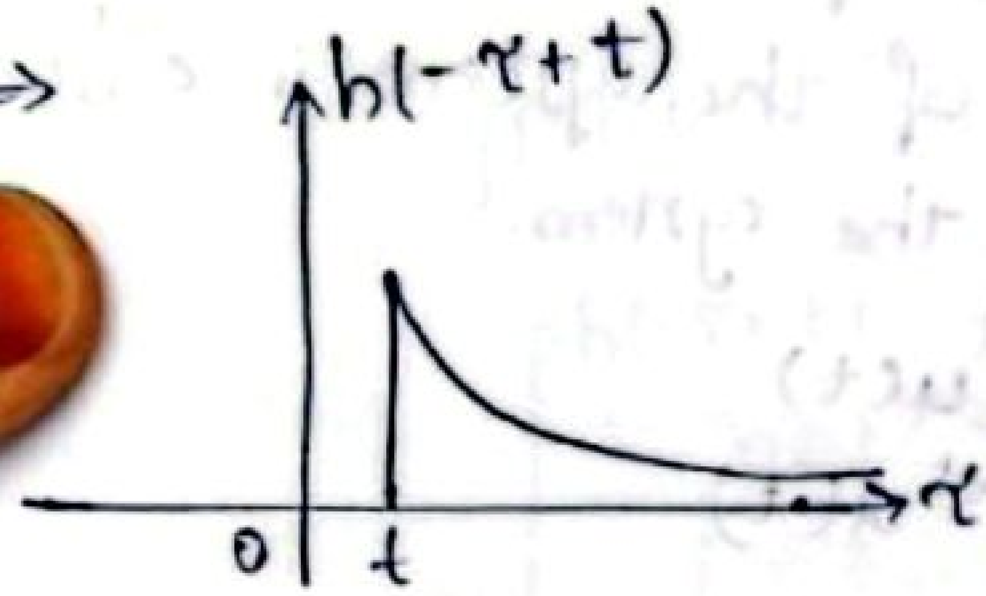


Exponentially decaying

$\Rightarrow$



$\Rightarrow$



$l_1 = 0$, $u_1 = \infty$

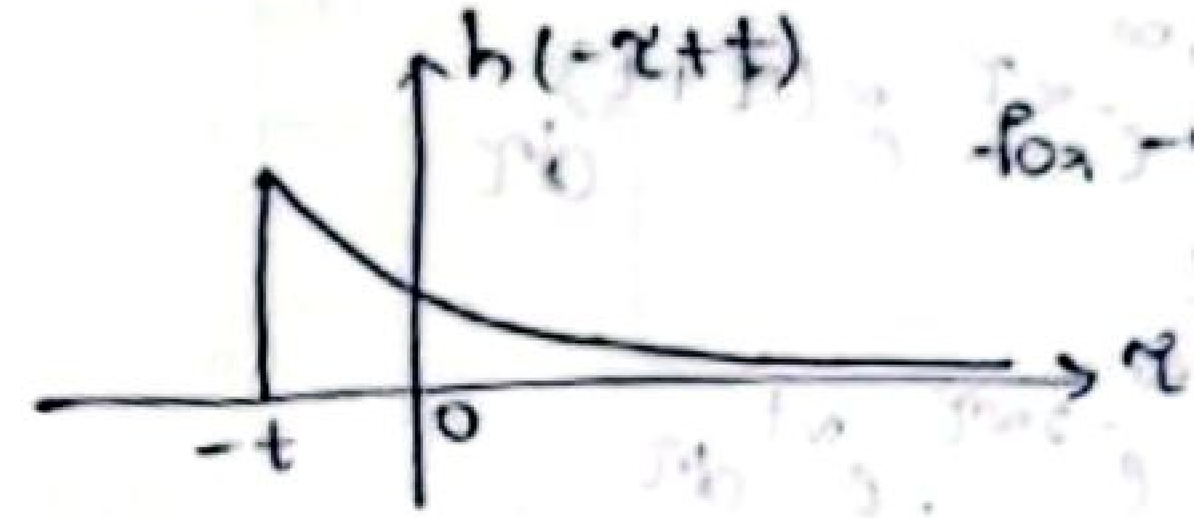
$l_2 = -\infty$, $u_2 = 0$

$l_1 + l_2$	$l_1 + u_2$	$l_2 + u_1$	$u_1 + u_2$
$-\infty$	0		∞

case (1)

for $-\infty < t < 0$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(-\tau+t)\, d\tau$$



for $-\infty < t < 0$

for $-\infty < t < 0$

$$y(t) = \int_0^{\infty} e^{-\alpha\tau} \cdot e^{\alpha(t-\tau)}\, d\tau$$

$$= \int_0^{\infty} e^{-\alpha\tau} \cdot e^{\alpha t} \cdot e^{-\alpha\tau}\, d\tau$$

$$= e^{\alpha t} \int_0^{\infty} e^{-2\alpha\tau}\, dt$$

$$= e^{\alpha t} \left[\frac{e^{-2\alpha\tau}}{-2\alpha} \right]_0^{\infty}$$

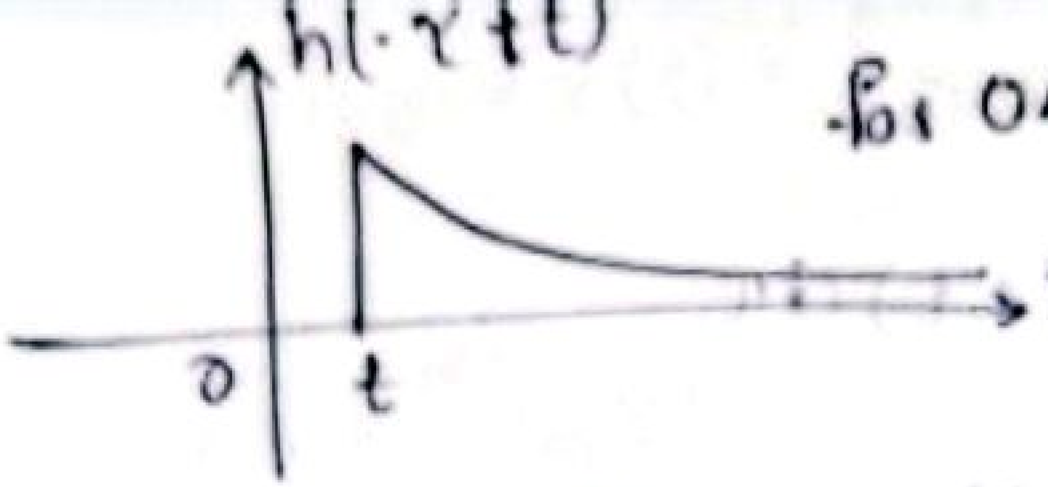
$$= \frac{e^{\alpha t}}{-2\alpha} [0 - 1]$$

$$y(t) = \frac{e^{\alpha t}}{2\alpha} \quad \text{for } -\infty < t < 0$$

case (2)

for $0 < t < \infty$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(-\tau+t)\, d\tau$$



for $0<t<\infty$

$$\therefore y(t) = \int_{t}^{\infty} e^{-\alpha\tau}\, e^{\alpha(-\tau+t)}\, d\tau$$

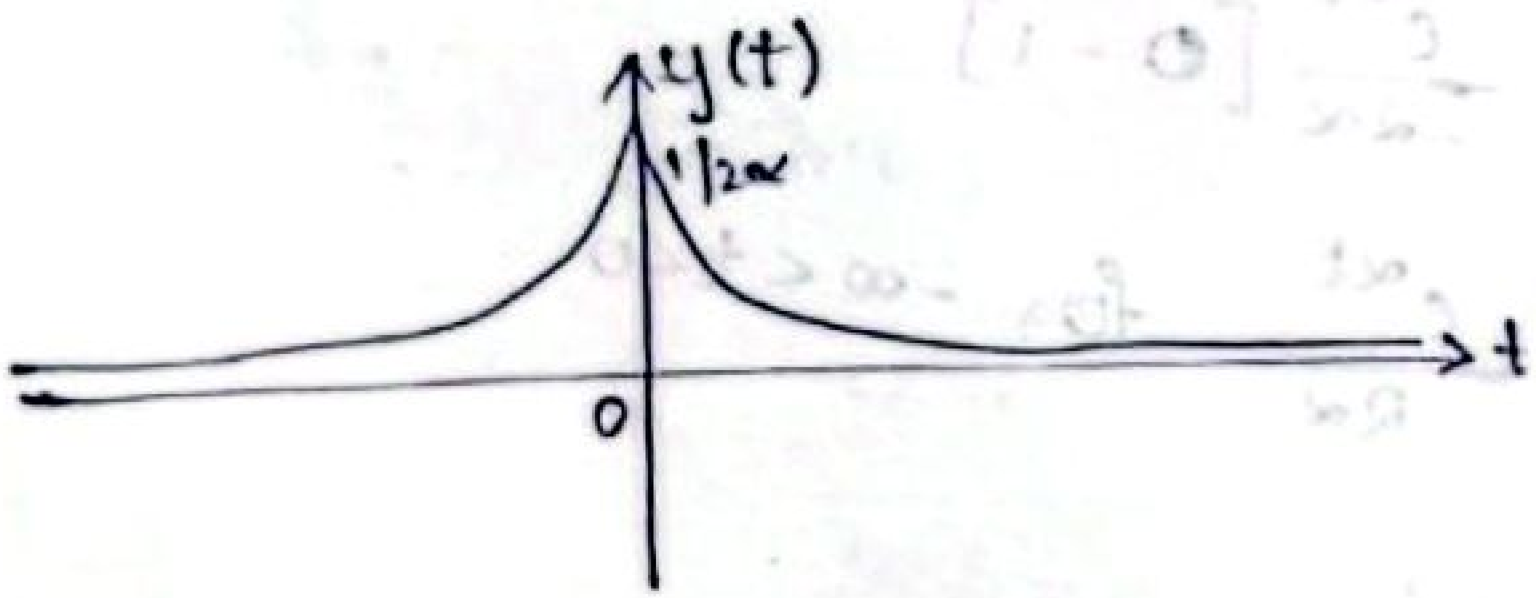
$$= \int_{t}^{\infty} e^{-2\alpha\tau} . e^{\alpha t}\, d\tau$$

$$= e^{\alpha t}\left[\frac{e^{-2\alpha\tau}}{-2\alpha}\right]_{t}^{\infty}$$

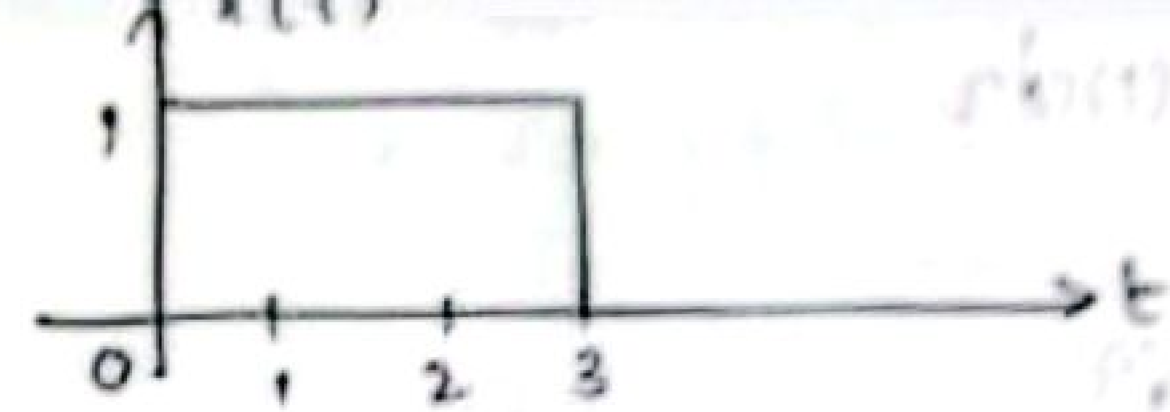
$$= \frac{e^{\alpha t}}{-2\alpha}\left[0 - e^{-2\alpha t}\right]$$

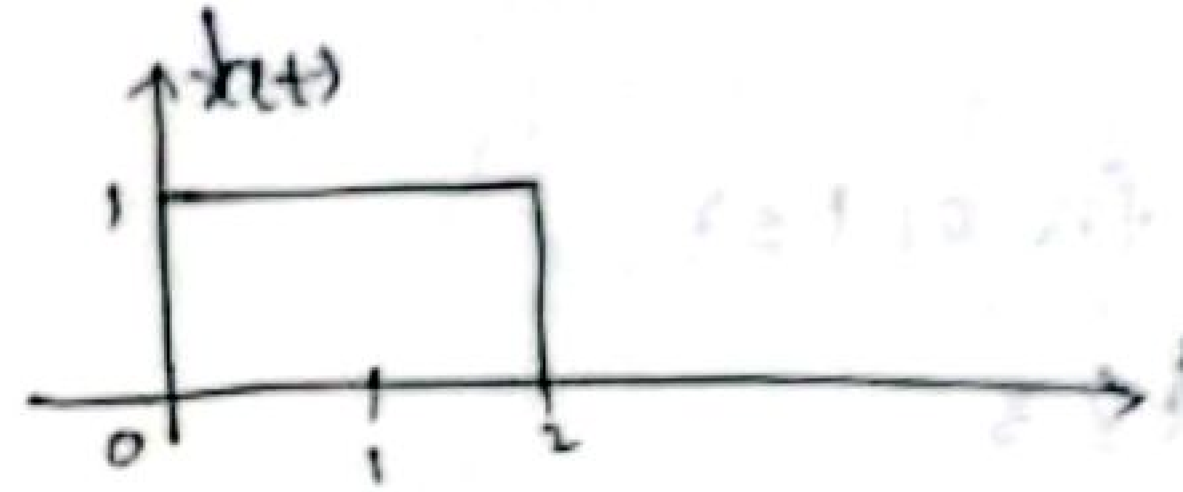
$$y(t) = \frac{e^{-\alpha t}}{2\alpha} \quad \text{for } 0<t<\infty$$

$$y(t) = \begin{cases} \frac{1}{2\alpha} e^{\alpha t} & \text{for } -\infty<t<0 \\ \frac{1}{2\alpha} e^{-\alpha t} & \text{for } 0<t<\infty \end{cases}$$

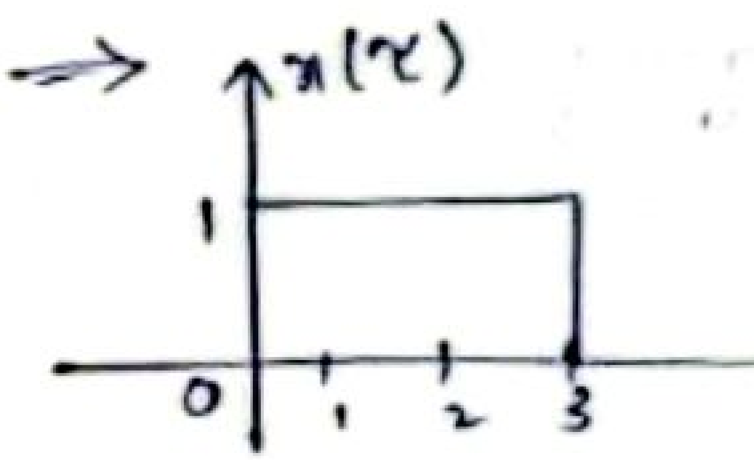


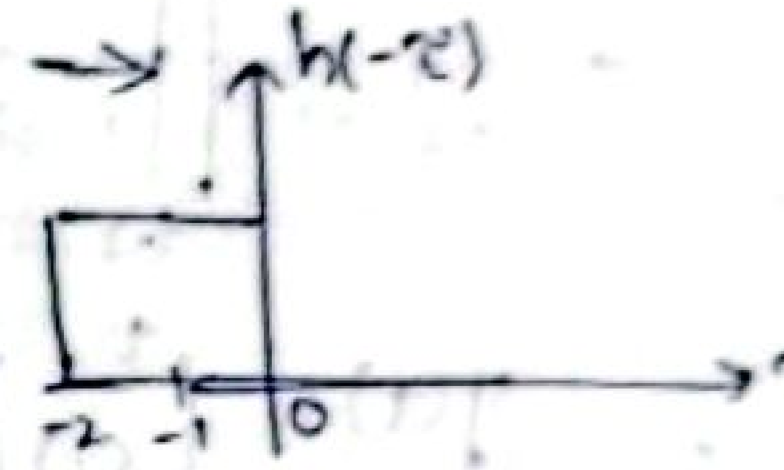
Q) Find the linear convolution b/w two following signals.

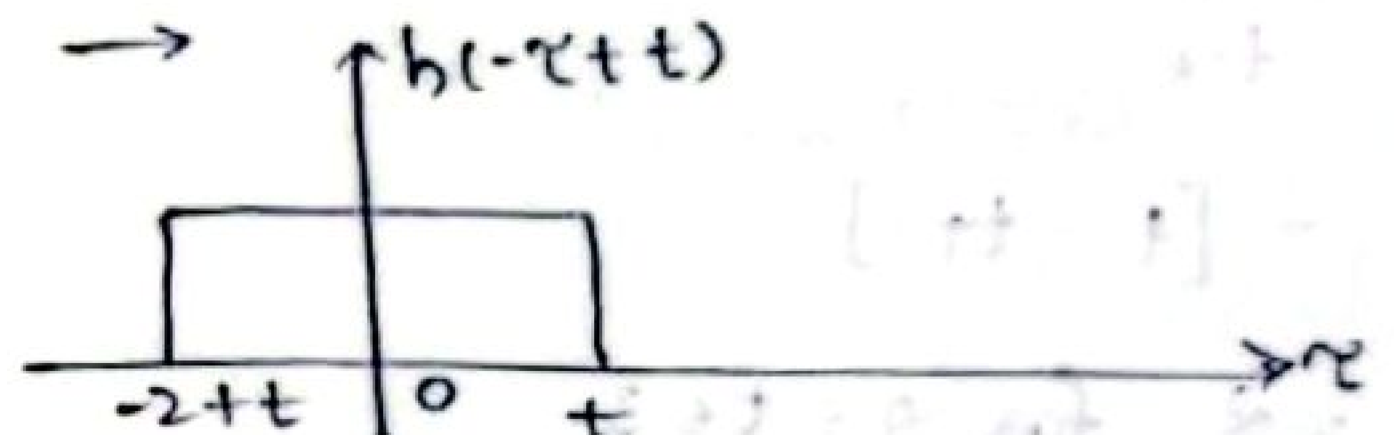




$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \quad \text{for } -\infty<t<\infty$$

→ 

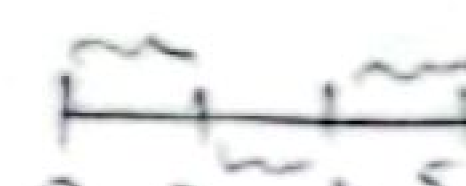
→ 

→ 

$l_1 = 0, \quad l_2 = 0$

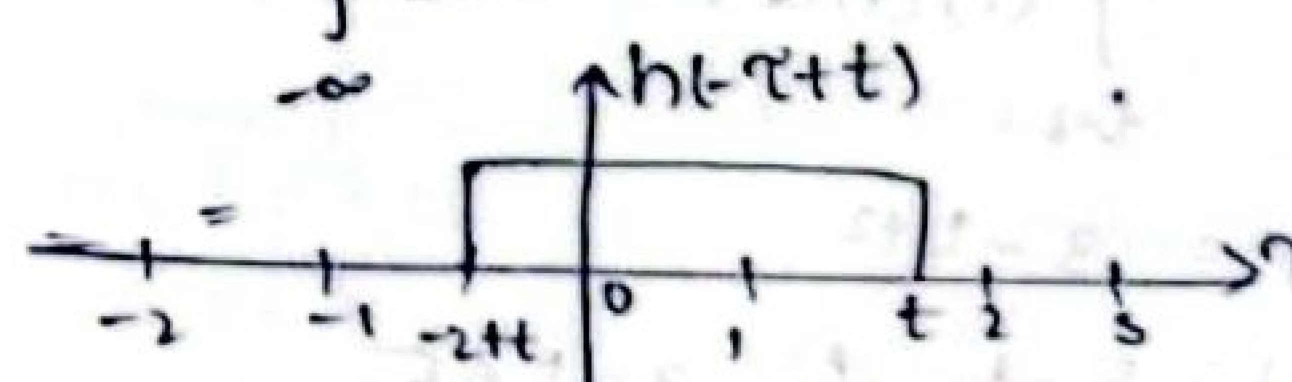
$U_1 = 3 \quad U_2 = 2$

l_1+l_2	l_1+U_2	l_2+U_2	U_1+U_2
0	3	2	5



case ① $0 \le t \le 2$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(-\tau+t)\, d\tau$$

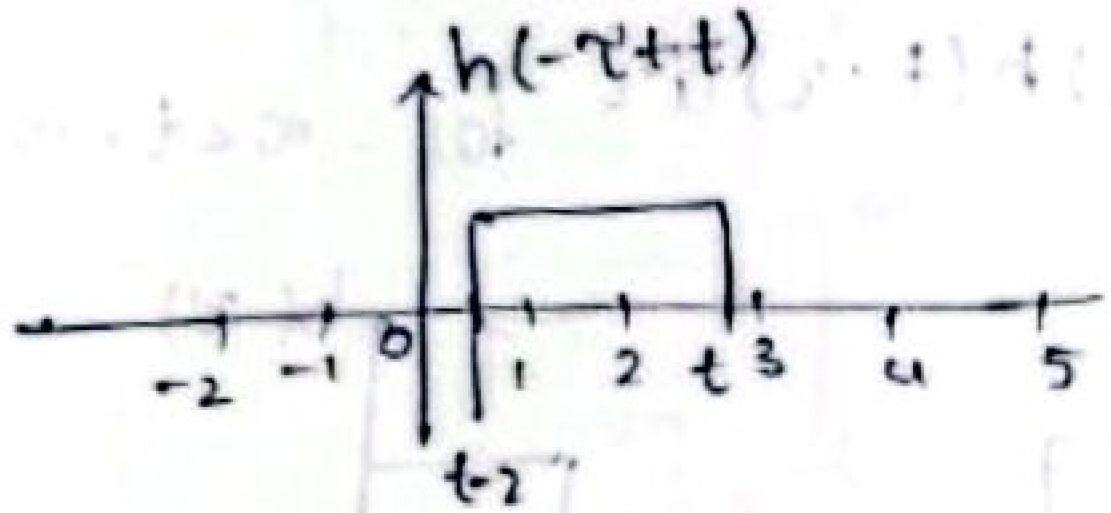


$$y(t) = \int_0^t (1)(1)\,d\tau$$

$$= \int_0^t d\tau$$

$y(t) = t$ for $0 \le t \le 2$

case ②:- $2 \le t \le 3$

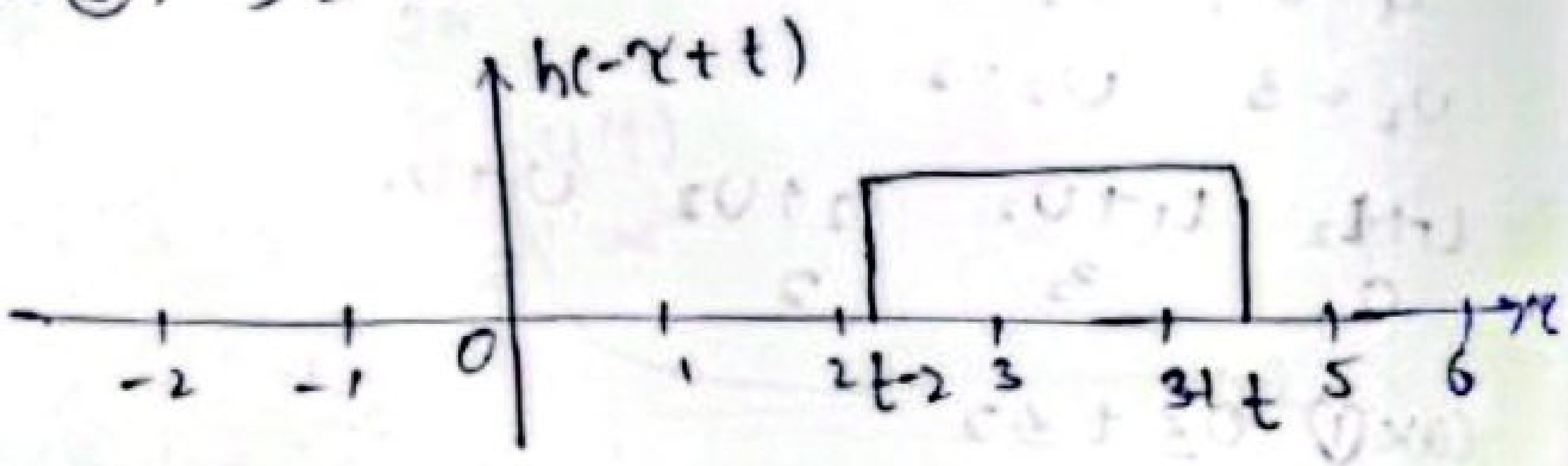


$$y(t) = \int_{t-2}^{t} (1)(1)\,d\tau$$

$$= [t - t + 2]$$

$y(t) = 2$ for $2 \le t \le 3$

case ③:- $3 \le t \le 5$

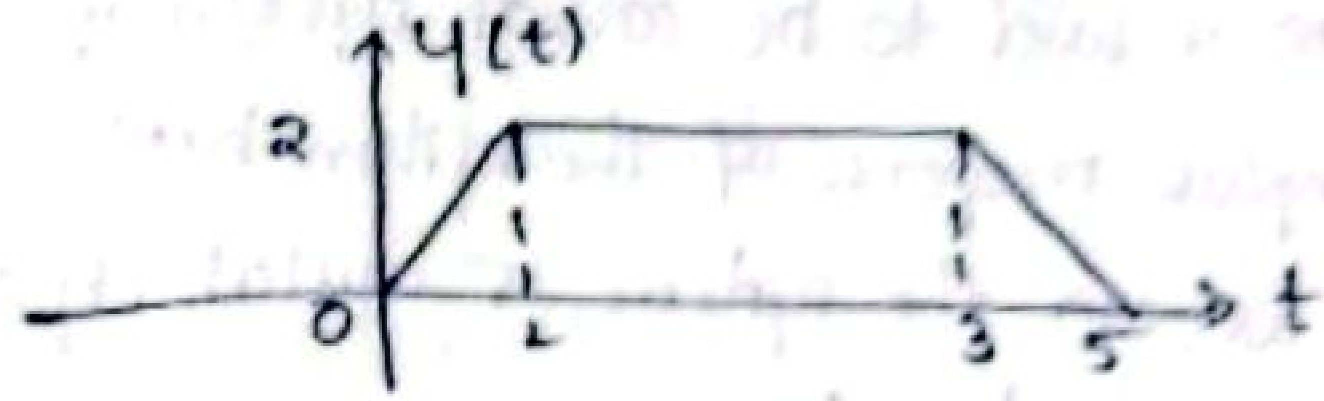


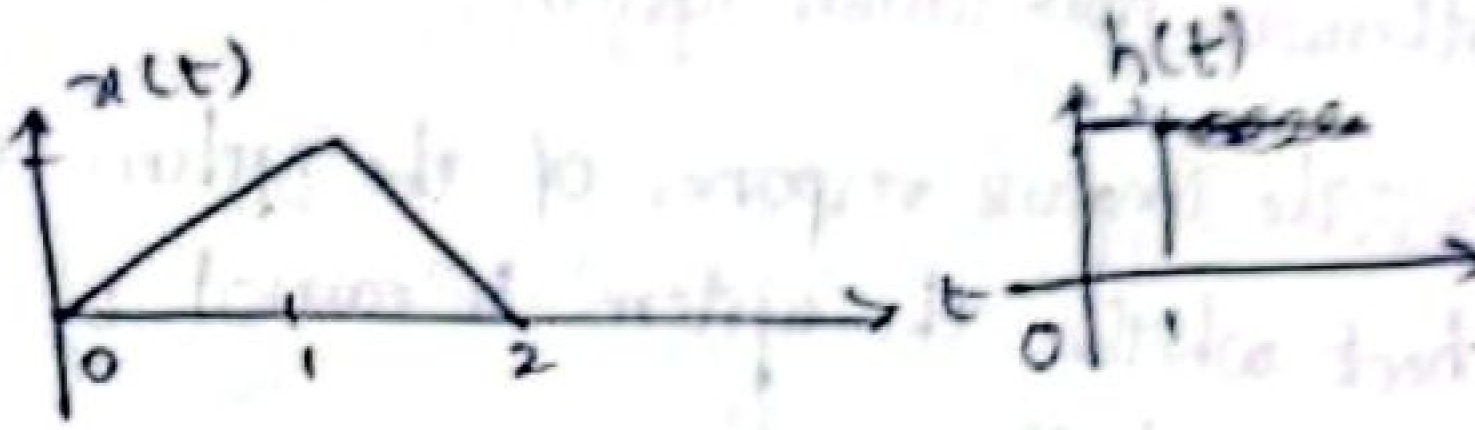
$$\therefore y(t) = \int_{t-2}^{3} (1)(1)\,d\tau$$

$$= 3 - t + 2$$

$y(t) = 5 - t$ for $3 \le t \le 5$

$$y(t) = \begin{cases} t & \text{for } \ldots \\ 2 & \text{for } 2 \le t \le 3 \\ 5-t & \text{for } 3 \le t \le 5 \end{cases}$$



a) 

sol:- $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$ for $-\infty < t < \infty$

case ①

$0 \le t \le 1$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

case ②

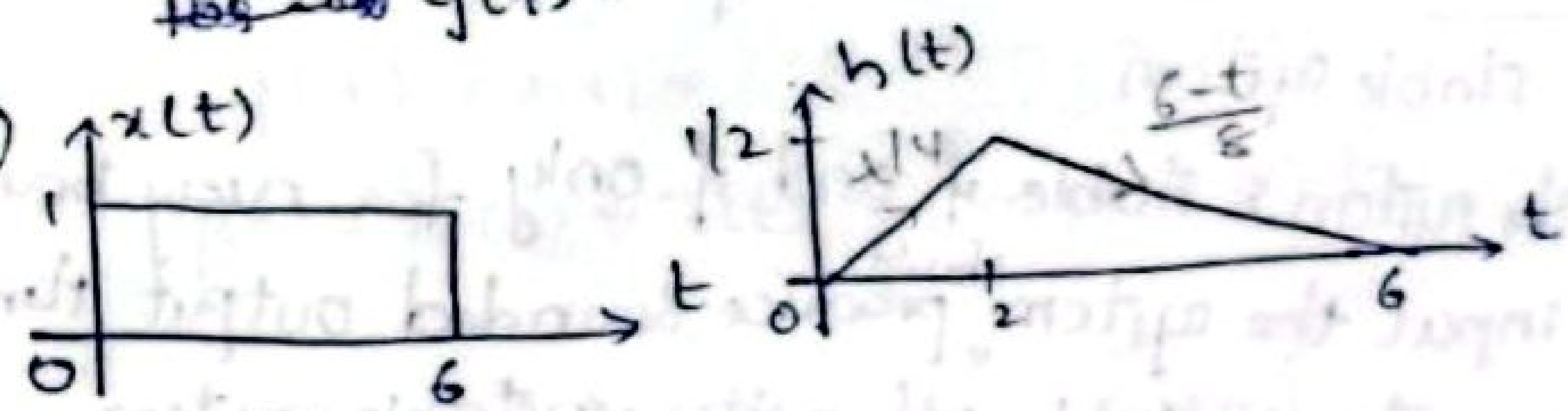
$1 \le t \le 2$

$y(t) =$

case ③

$2 \le t \le 3$

$y(t) =$

(b) 

sol:- cases:-

$0 \le t \le 2$

$2 \le t \le 6$

$6 < t < 8$

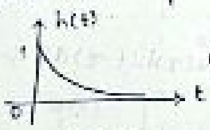
$8 \le t \le 13$

Q:- Causal vs non-causal system

If a system is said to be causal system if and if only the impulse response of the system $h(t) = 0$ for $t < 0$. Then we can the system is causal system otherwise non-causal system.

Ex:① The impulse response of the system is $h(t) = e^{-t}u(t)$ check whether the system is causal or non causal

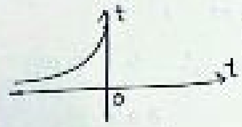
So:



causal system

② If $h(t) = e^{2t}u(-t)$

$h(t)$ is non zero for $t < 0$



noncausal system

Stable vs unstable system

Method 1: BIBO (Bounded Input Bounded output) stable system

→ The system is stable if and if only for every bounded input the system should produce bounded output then only system is stable otherwise unstable system

Ex:

i/p $x(t)$ → [LTI system] → o/p $y(t)$

$|x(t)| \le M_x < \infty \ \forall\ t$ — Bounded Input

$|y(t)| \le M_y < \infty \ \forall\ t$ — Bounded output

③ check whether the following system is stable or unstable $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$

BI $x(t) = u(t)$

$y(t) = \int_{-\infty}^{t} u(\tau)d\tau$

$y(t) = \int_{-\infty}^{t} u(\tau)d\tau = \int_{0}^{t} 1 \cdot d\tau = t$

$\therefore y(t) = t$ → unbounded signal

$\therefore$ unstable system

Method 2: stability of the system Based on the impulse response.

* BIBO stable system

BI: $|x(t)| \le M_x < \infty$

BO: $|y(t)| \le M_y < \infty$

$x(t)$ i/p → [$h(t)$ LTI system] → $y(t)$ o/p

$y(t) = x(t) * h(t) = h(t) * x(t)$

$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$

$\therefore |y(t)| = \left| \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \right|$

$= \left| \int_{-\infty}^{\infty} h(\tau)x(t-\tau) \right| d\tau$

b] $\Rightarrow |x(t)| \le M_x$

similarly $|x(t-\tau)| \le M_x$

$$|y(t)| \le M_x \int_{-\infty}^{\infty} |h(\tau)| d\tau$$

BO $\rightarrow |y(t)| \le M_y < \infty$

This is possible if $\int_{-\infty}^{\infty} |h(\tau) d\tau| < \infty$

Then the system is stable

where $h(t)$ is the implulse response of LTI system

Q) check whether the following system is stable or unstable. $h(t) = e^{-3t} u(t)$.

sol: The implulse response of the system $h(t) = e^{-3t} u(t)$

$$\because \int_{-\infty}^{\infty} |h(t)| dt = \int_{0}^{\infty} e^{-3t} dt = 1/3 < \infty$$

$\therefore$ stable system

Q) $h(t) = e^{3t} u(t)$

$$\therefore \int_{-\infty}^{\infty} |h(t)| dt = \int_{0}^{\infty} e^{3t} dt = \frac{1}{3}(\infty - 1)$$

$$= \infty$$

$\therefore$ unstable system

step Response of the system

$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$

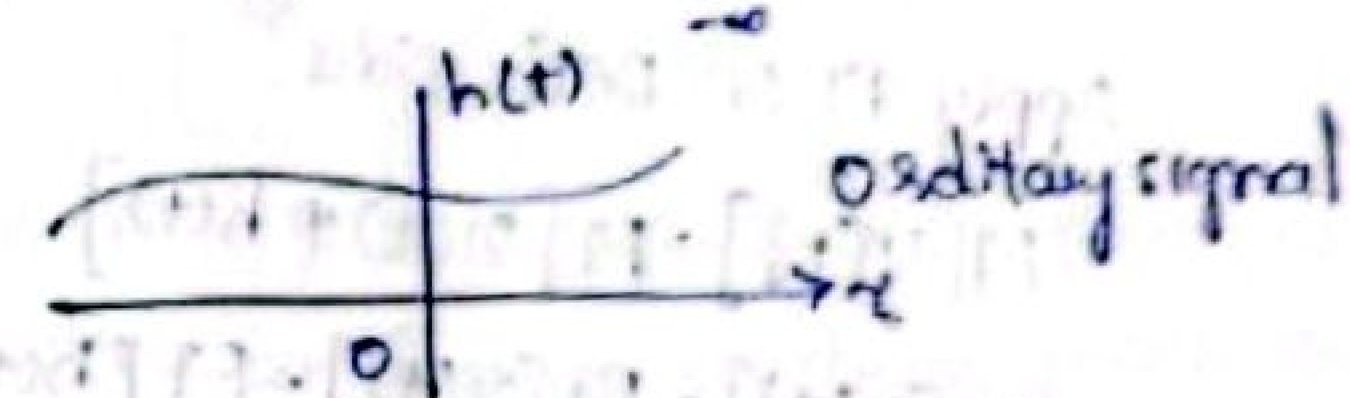
LTI system

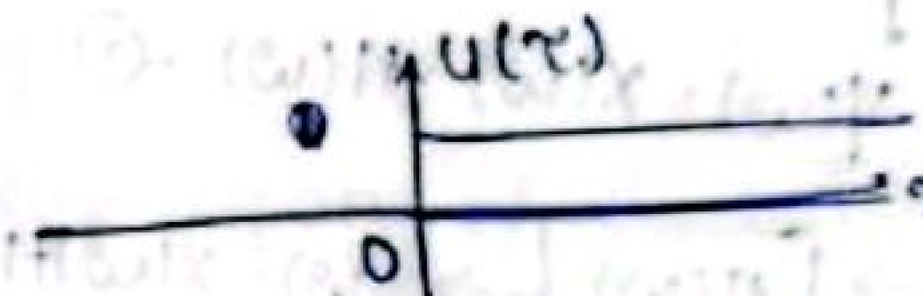
if i/p to the LTI system is unit step system the corresponding response of the system ~~of the system~~ is know as step response of the system

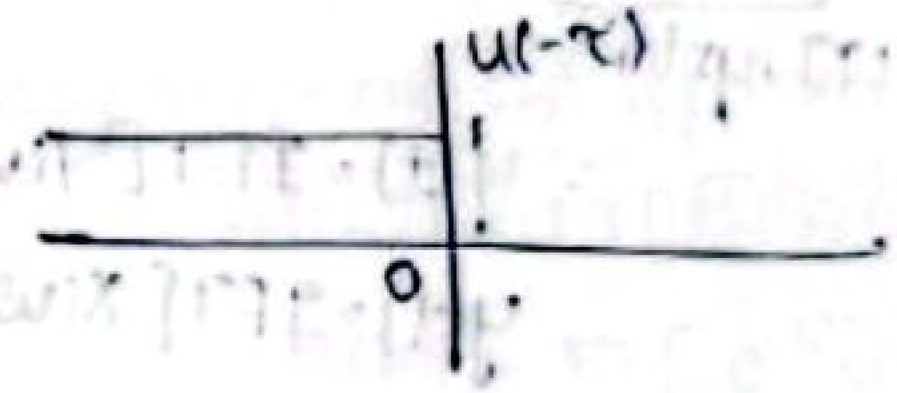
$$y(t) = x(t) * h(t)$$

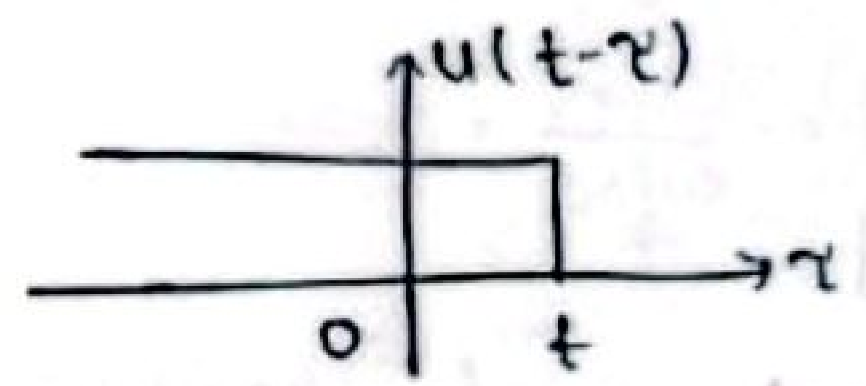
$$s(t) = u(t) * h(t) = h(t) * u(t)$$

$$s(t) = h(t) * u(t) = \int_{-\infty}^{\infty} h(\tau) u(t-\tau) d\tau$$









$$s(t) = \int_{-\infty}^{t} h(\tau)(1) d\tau$$

$$s(t) = \int_{-\infty}^{t} h(\tau) d\tau$$

$\because s(t) =$ step response of the system

$h(t) =$ Implulse response of the system

$$\boxed{h(t) = \frac{ds(t)}{dt}}$$

method ① $x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = x(t) * h(t)$
i/p LTI system o/p

Apply FT on Both sides

$$FT[y(t)] = FT[x(t) * h(t)]$$

$$FT[y(t)] = FT[x(t)] \cdot FT[h(t)]$$

$$Y(\omega) = X(\omega) \cdot H(\omega) \quad -①$$

$X(\omega) \longrightarrow \boxed{H(\omega)} \longrightarrow Y(\omega) = X(\omega)H(\omega)$
i/p LTI system

method ② $\because y(t) = IFT[Y(\omega)]$

$$y(t) = IFT[X(\omega)H(\omega)]$$

→ $H(\omega) = FT[h(t)]$

$h(t) = IFT[H(\omega)]$

$\therefore h(t) \xleftrightarrow{FT} H(\omega)$

Impluse response of the system — frequency response of the system

from Eqⁿ ① $\boxed{H(\omega) = \frac{Y(\omega)}{X(\omega)}}$

$$H(\omega) = FT[h(t)] = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} \, dt$$

$H(\omega)$ is alway complex

$$\therefore H(\omega) = |H(\omega)| e^{j\theta(\omega)}$$

where $\theta(\omega) = \angle H(\omega)$

→ The plot b/w $|H(\omega)|$ vs ω is called magnitude spectrum

→ The plot b/w $\theta(\omega)$ vs ω is called phase spectrum

→ The combination of this two is called frequency spectrum.

→ If $h(t)$ is real, the magnitude spectrum posses Even symmetry and phase spectrum posses odd symmetry.

① The impluse response of the system $h(t) = e^{-3t}u(t)$ find the frequency response of the system

frequency response of system which is

$$H(\omega) = FT[h(t)]$$

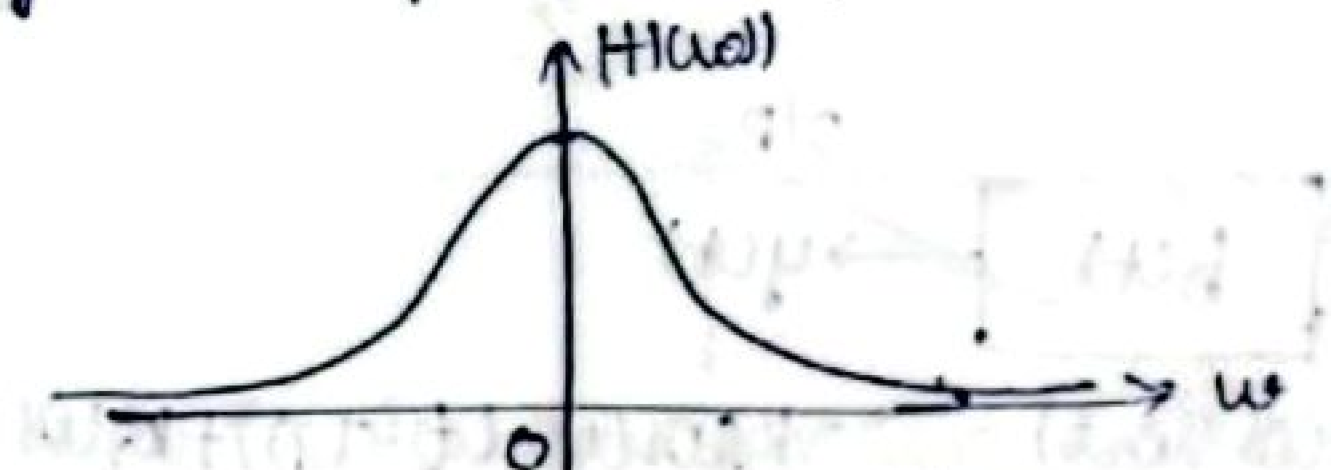
$$H(\omega) = FT[e^{-3t}u(t)]$$

$$H(\omega) = \frac{1}{3+j\omega} \rightarrow \text{frequency Response of the system}$$

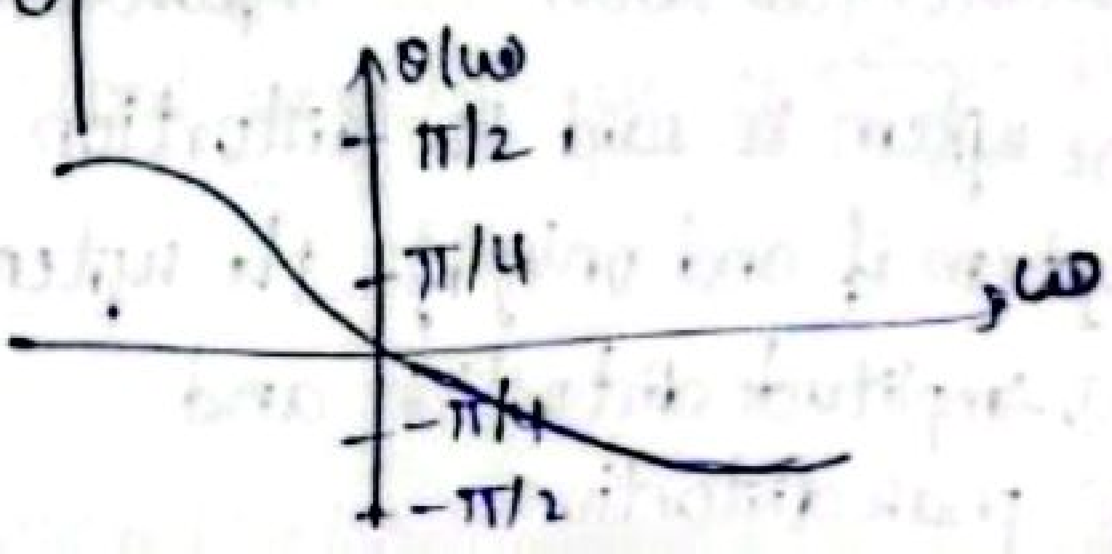
$$\therefore |H(\omega)| = \frac{1}{\sqrt{9+\omega^2}} = \text{magnitude function}$$

$$\theta(\omega) = \angle H(\omega) = -\tan^{-1}\left(\frac{\omega}{3}\right) \rightarrow \text{phase function}$$

Magnitude response



phase spectrum



⑤ Frequency response of the system is $H(\omega) = \frac{1}{1+\omega^2}$
find the impluse response of the system

Amplitude distortion (Magnitude distortion)

i/p $x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$ o/p

$x(t) = \cos(\omega_1 t) + \cos(\omega_2 t)$

$y(t) = k_1 \cos(\omega_1 t) + k_2 \cos(\omega_2 t)$

$\omega_1 \neq \omega_2$

→ if $k_1 \neq k_2$
The system produces / provides Amplitude distortion

phase distortion (Delay distortion)

i/p $x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$ o/p

$x(t) = \cos(\omega_1 t) + \cos(\omega_2 t)$ $\quad$ $y(t) = \cos(\omega_1(t-t_1)) + \cos(\omega_2(t-t_2))$

$\omega_1 \neq \omega_2$ $\quad$ if $t_1 \neq t_2$

The system provides phase distortion.

distortionless transmission systems:-

i/p $x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$ o/p

$x(t) = \cos(\omega_1 t) + \cos(\omega_2 t)$ $\quad$ $y(t) = k_1\cos(\omega_1(t-t_1)) + k_2\cos(\omega_2(t-t_2))$

The system is said to distortion less trasmission system if and only if the system is free from
① Amplitude distortion and
② phase distortion.

$\therefore k_1 = k_2$
$t_1 = t_2 = t_0$

$\therefore y(t) = k[\cos(\omega_1(t-t_0)) + \cos(\omega_2(t-t_0))]$

$\therefore y(t) = kx(t-t_0)$

Apply FT on both sides

$\because FT[y(t)] = FT[kx(t-t_0)]$

$Y(\omega) = kX(\omega)e^{-j\omega t_0}$

$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{kX(\omega)e^{-j\omega t_0}}{X(\omega)}$

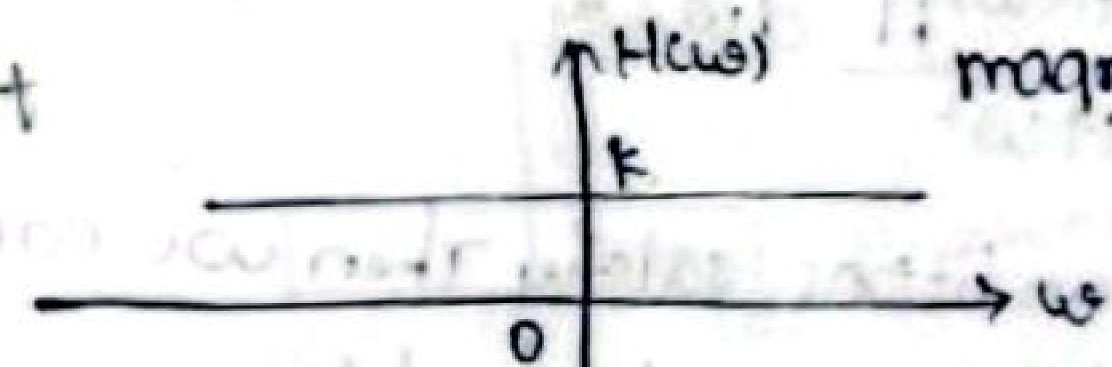
$\boxed{\therefore H(\omega) = ke^{-j\omega t_0}}$

$H(\omega) = |H(\omega)|e^{j\theta(\omega)} = k \cdot e^{-j\omega t_0}$

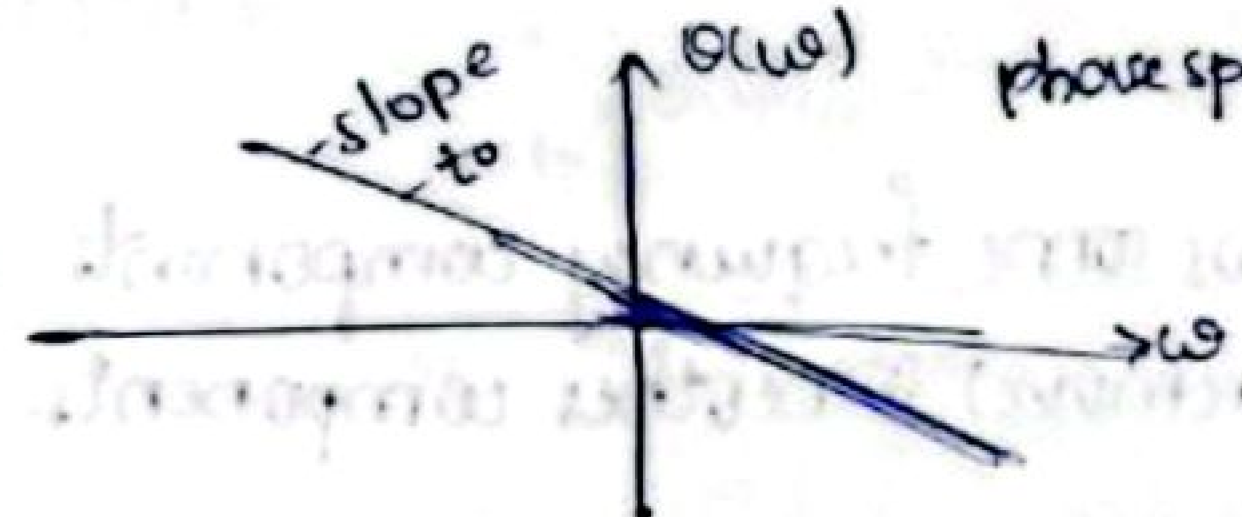
$\therefore |H(\omega)| = k$

$\theta(\omega) = \angle H(\omega) = -\omega t_0$

constant



linear



* for distortionless transmission system, magnitude spectrum is constant and phase spectrum is linear

→ This criteral is used to check whether the system is physically reliazable (or) not.

① → If the system is causal and stable then the system is said to be physically reliazable.

→ The implus response = 0 i.e, $h(t) = 0$ for $t < 0$ then the system is causal system

where $h(t)$ = implus response of system

→ $\int_{-\infty}^{\infty} |h(t)| dt < \infty$ this is stable condition

②

Paley wiener critertion

① $\int_{-\infty}^{\infty} |H(\omega)|^2 d\omega < \infty$

$H(\omega)$ is frequency response of the system

② $\int_{-\infty}^{\infty} \frac{|\ln |H(\omega)||}{1+\omega^2} d\omega < \infty$

If this ② conditions satefy then we can say system is physically reliazable.

<u>Filters:</u>

filters allows some frequency components and attenuate (remove) some other components.

i/p $x(t)$ → $\boxed{H(\omega)}$ → o/p

filter

① $y(t) = x(t) * h(t)$

② $Y(\omega) = X(\omega) H(\omega)$

$y(t) = \mathrm{IFT}[Y(\omega)] = \mathrm{IFT}[X(\omega) H(\omega)]$

① ideal low-pass filter (ILPF)

ideal low-pass filter, allows low frequency components and remove high-frequency components. The filter is characteristics by its frequency response

$H(\omega) = |H(\omega)| e^{j\theta(\omega)}$

where $\theta(\omega) = \angle H(\omega)$

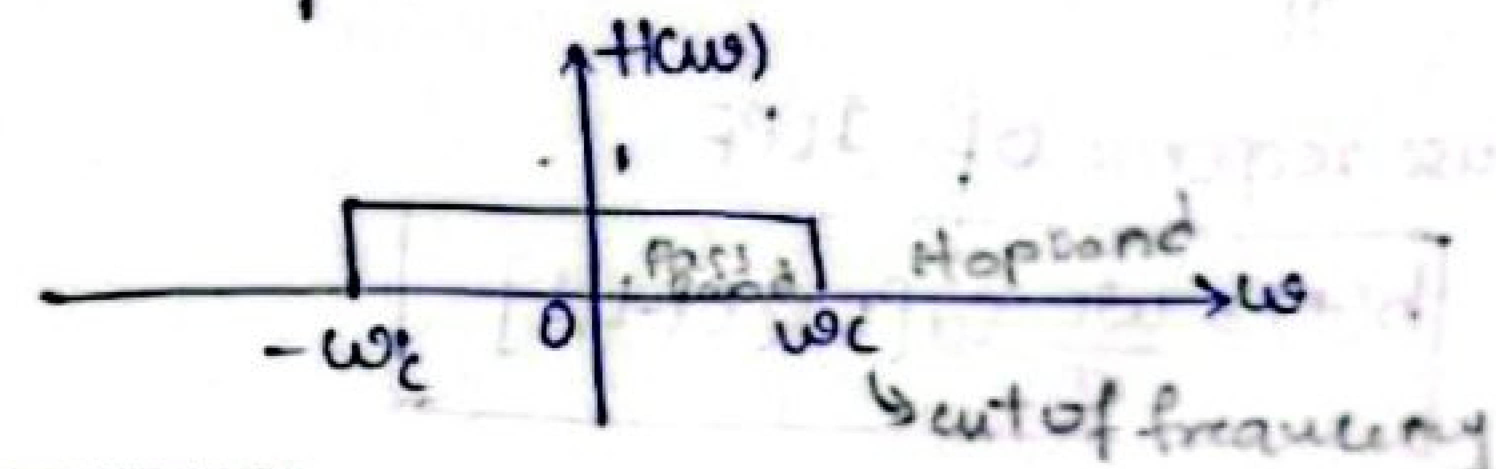
for ILPF magnitude function is

$$H(\omega) = \begin{cases} 1 & \text{for } -\omega_c \le \omega \le \omega_c \\ 0 & \text{otherwise} \end{cases}$$

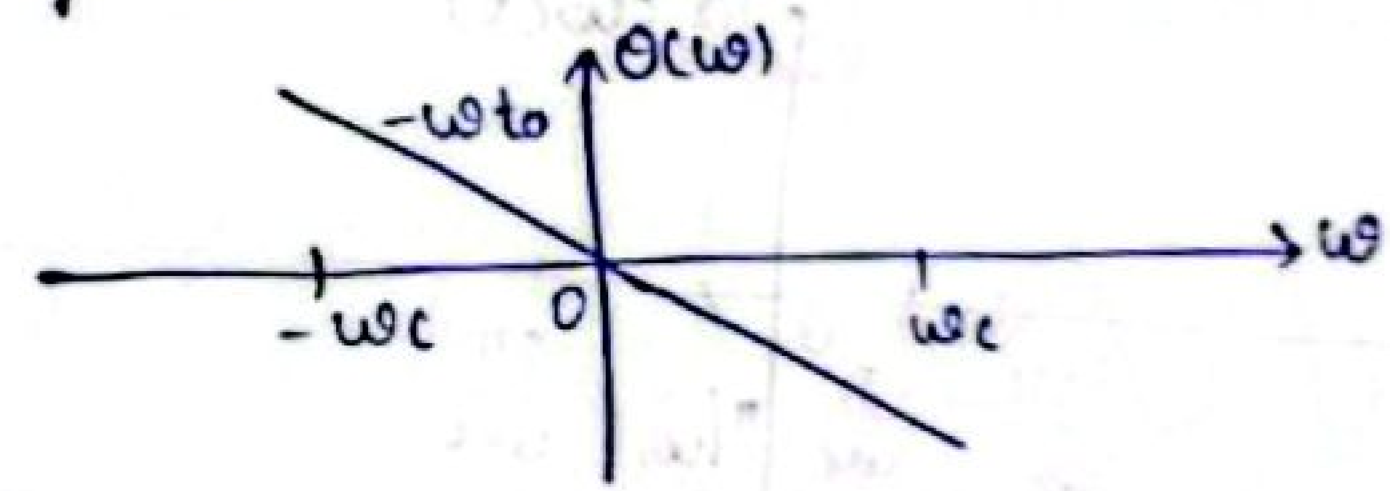
and phasefunction is

$$\theta(\omega) = \begin{cases} -\omega t_0 & \text{for } -\omega_c \le \omega \le \omega_c \\ 0 & \text{otherwise.} \end{cases}$$

Magnitude response



phase response



frequency response :

$H(\omega) = |H(\omega)| e^{j\theta(\omega)}$

$H(\omega) = \mathrm{rect}\left(\frac{\omega}{2\omega_c}\right) e^{-j\omega t_0}$

The impluse response of ILPF

$h(t) = IFT[H(\omega)]$

$\downarrow$ impluse response, $\downarrow$ frequency response

we know that $sa(Bt) \xleftrightarrow{FT} \frac{\pi}{B} rect\left(\frac{\omega}{2B}\right)$

let $B = \omega_c$

$\therefore sa(\omega_c t) \xleftrightarrow{FT} \frac{\pi}{\omega_c} rect\left(\frac{\omega}{2\omega_c}\right)$

let $z(t) = \frac{\omega_c}{\pi} sa(\omega_c t) \xleftrightarrow{FT} rect\left(\frac{\omega}{2\omega_c}\right) = z(\omega)$

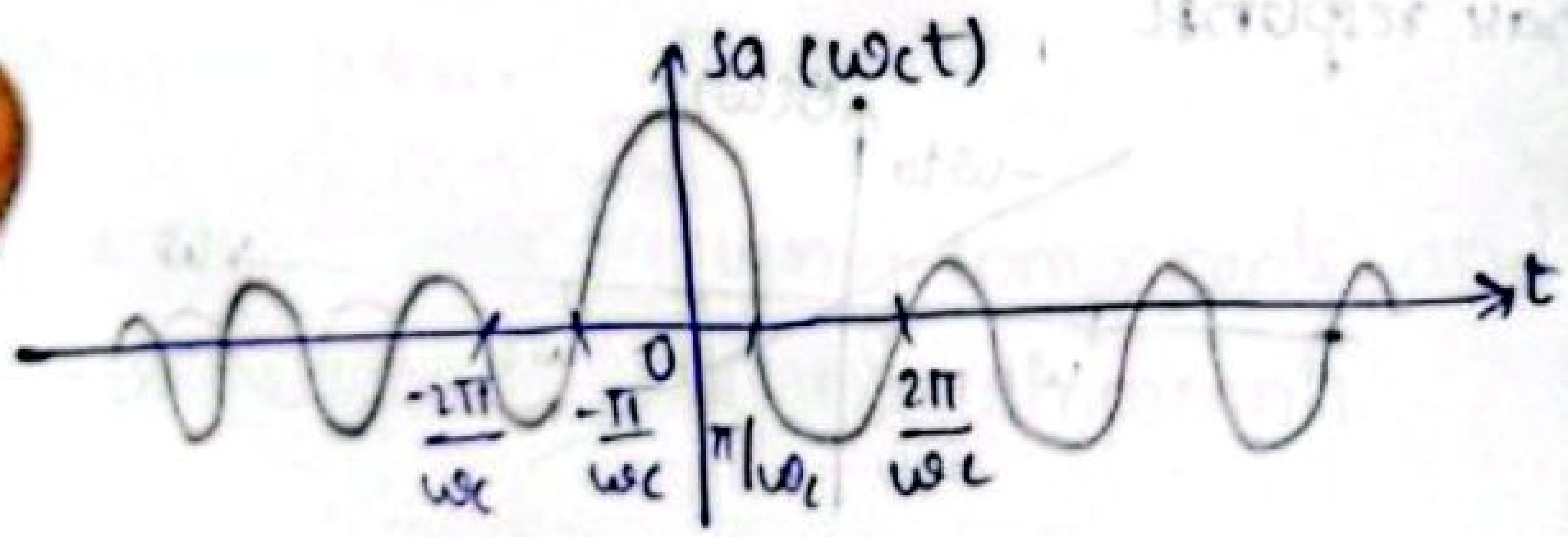
$z(t-t_0) \xleftrightarrow{FT} z(\omega) e^{-j\omega t_0}$

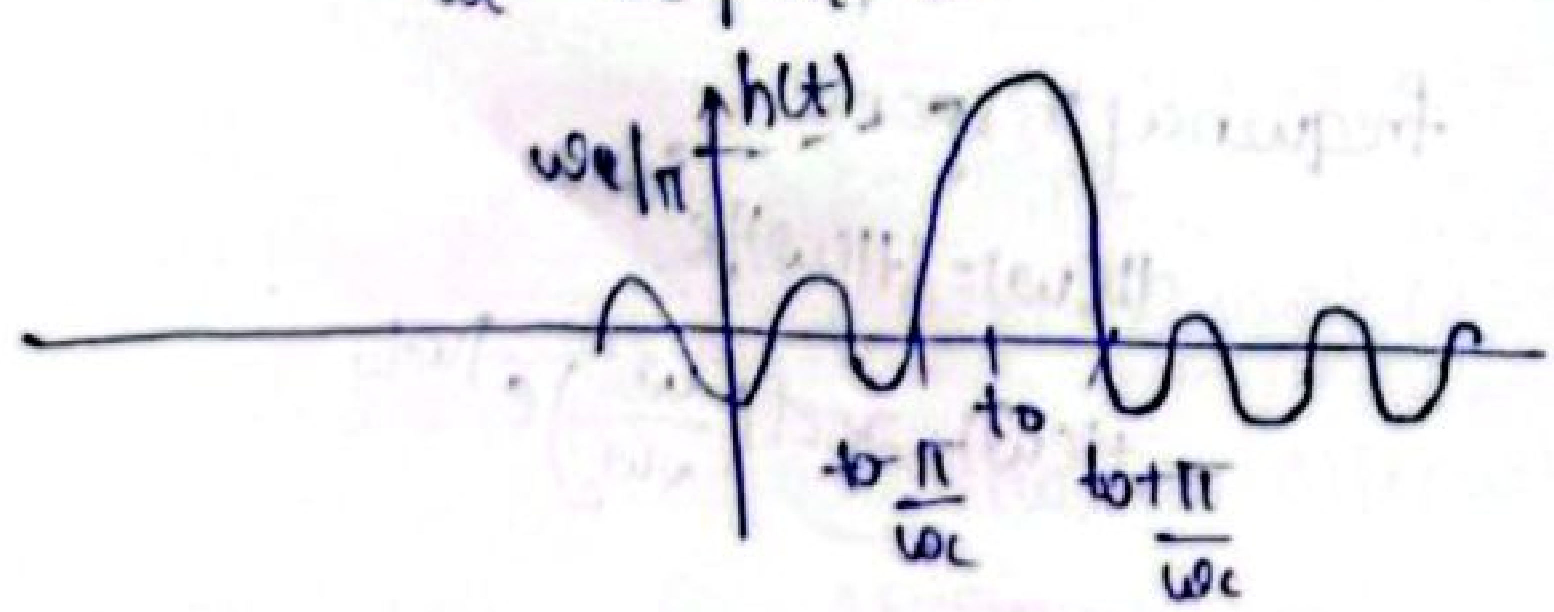
$\therefore \frac{\omega_c}{\pi} sa[\omega_c(t-t_0)] \xleftrightarrow{FT} rect\left(\frac{\omega}{2\omega_c}\right) e^{-j\omega t_0}$

$h(t) = \frac{\omega_c}{\pi} sa(\omega_c(t-t_0)] \xleftrightarrow{FT} H(\omega) = rect\left(\frac{\omega}{2\omega_c}\right) e^{-j\omega t_0}$

impluse response of ILPF

$$\boxed{h(t) = \frac{\omega_c}{\pi} sa[\omega_c(t-t_0)]}$$



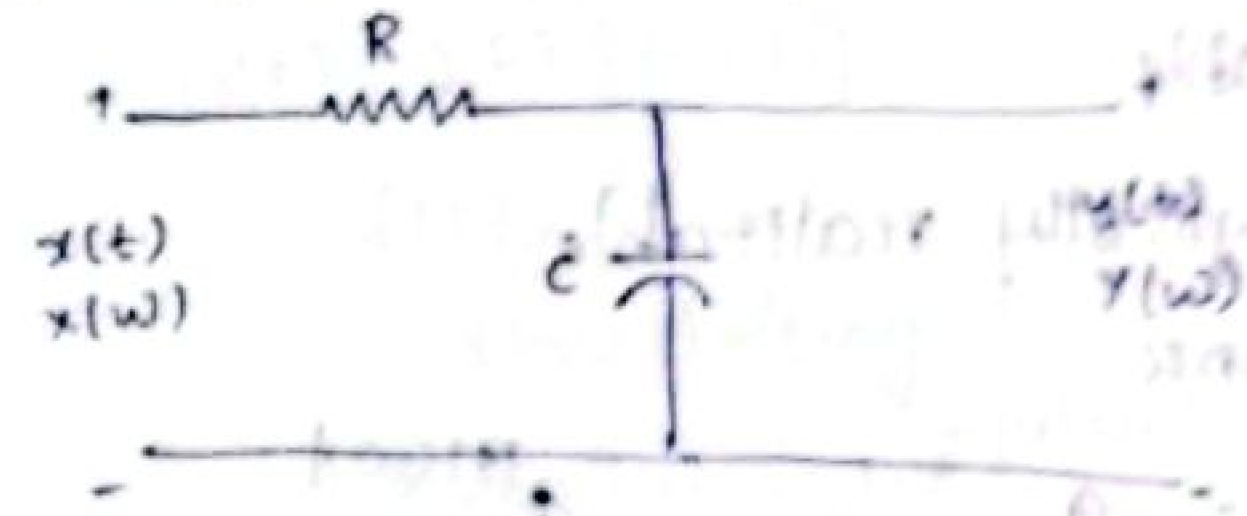


when ...

filter is non causal filter

$\therefore$ filter is not physically realizable.

② Practical filters (RC low pass filters)



By using voltage division Rule

$$Y(\omega) = \left(\frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}}\right) X(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{1 + Rj\omega C}$$

$$H(\omega) = \frac{\frac{1}{RC}}{\frac{1}{RC} + j\omega}$$

let $a = 1/RC$

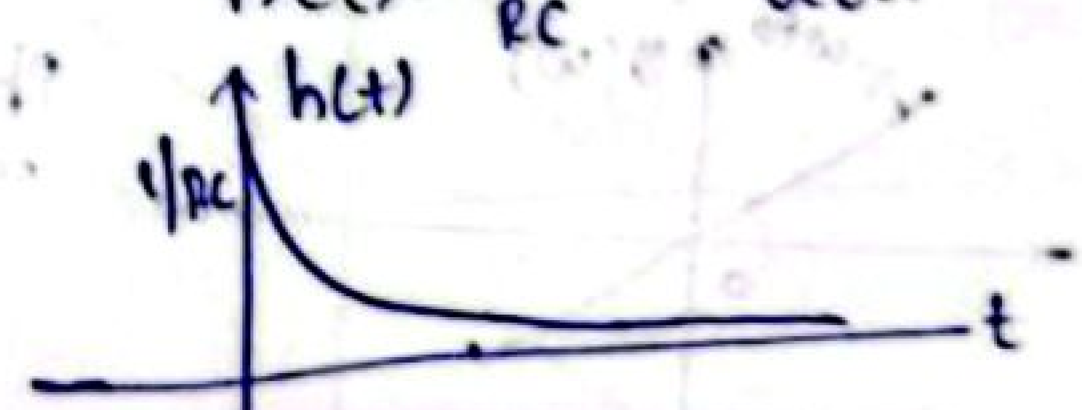
$$H(\omega) = \frac{a}{a + j\omega} \quad \text{(frequency response)}$$

Impluse response

$h(t) = IFT[H(\omega)]$

$h(t) = a \cdot e^{-at} u(t)$

$h(t) = \frac{1}{RC} e^{-t/RC} u(t)$



$h(t) = 0$ for $t < 0$

it is a causal signal

and $\int_{-\infty}^{\infty} |h(t)| dt < \infty$

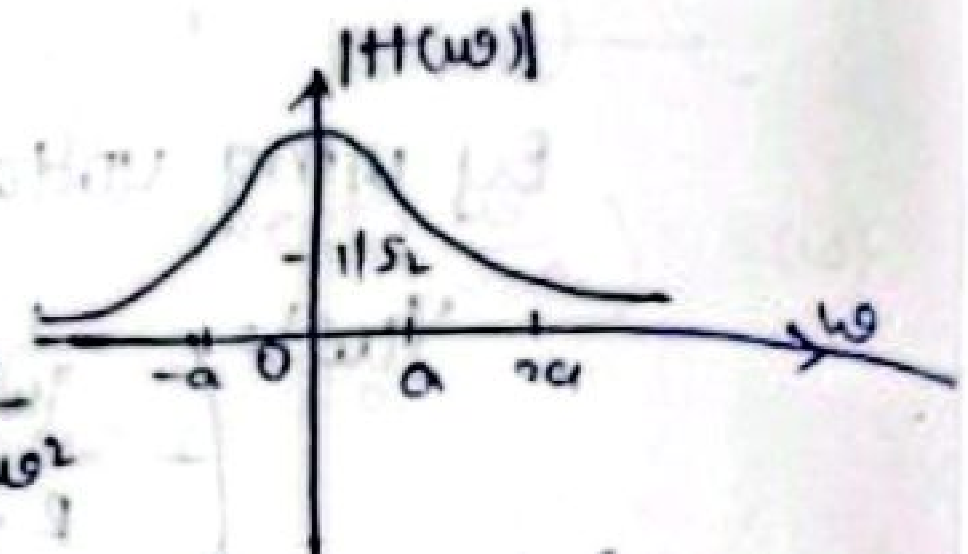
it is a stable

$\therefore$ It is physically realizable

Magnitude response

$H(\omega) = \frac{a}{a + j\omega}$

$|H(\omega)| = \frac{a}{\sqrt{a^2 + \omega^2}}$



phase response

3dB - Bandwidth = a

$\theta(\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$

Allpass filter:-

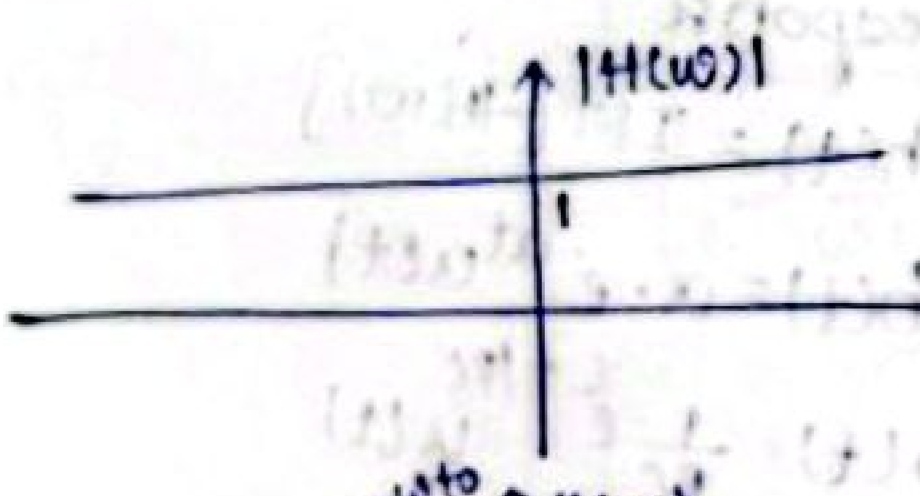
- Allpass filter passes all frequency components

i.e, magnitude response, $|H(\omega)| = 1 \ \forall \omega$

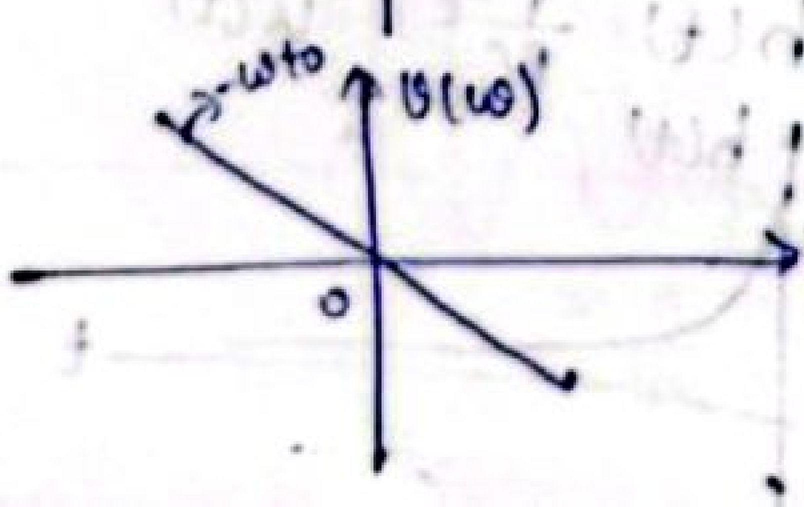
and phase response $\theta(\omega) = \angle H(\omega) = -\omega t_0$

$x(t) \longrightarrow \boxed{H(\omega)} \xrightarrow{(o/p)} y(\omega) = X(\omega)H(\omega)$

$X(\omega)$ filter



magnitude response



Phase response

Frequency resp.

$H(\omega) = |H(\omega)| e^{j\theta(\omega)}$

$H(\omega) = 1 \cdot e^{-j\omega t_0}$

Impulse response

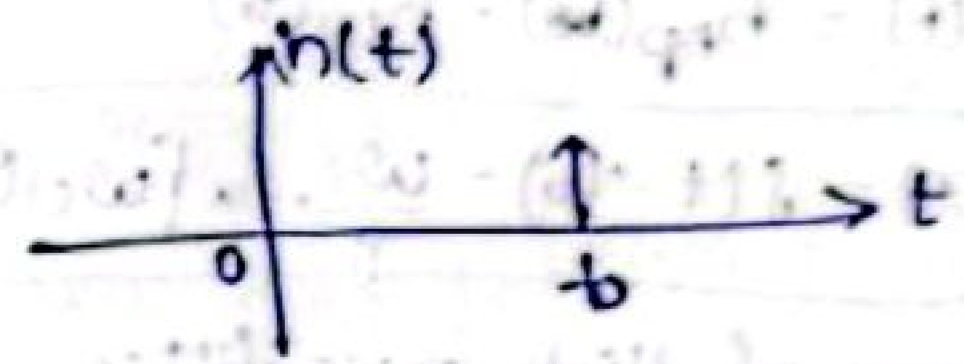
$h(t) = IFT[H(\omega)]$

$\delta(t) \xleftrightarrow{FT} 1$

time shifting

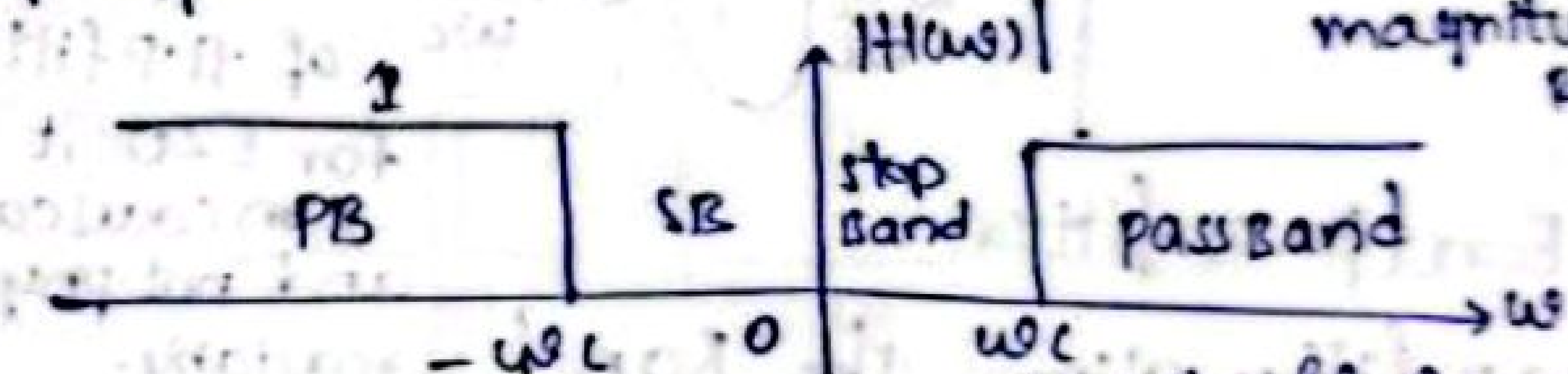
$\delta(t - T_0) = 1 \cdot e^{-j\omega t_0}$

$h(t) = \delta(t - t_0)$



Ideal highpass filter:-

This filter allow high frequency components of the input and removes low-frequency components of input



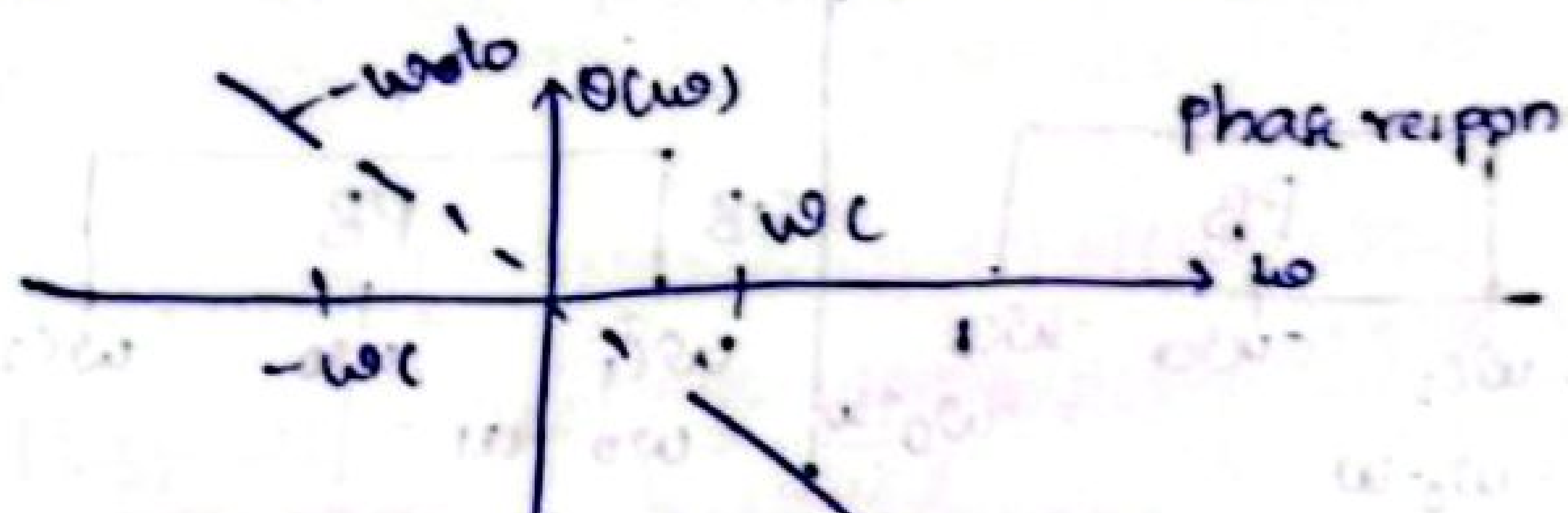
magnitude response

ωc is cut-off frequency

Note:-

3dB - Bandwidth:-

The frequency at which the magnitude value of the filter falls down to $1/\sqrt{2}$ times of its max value



Phase response

Frequency response

$$H(\omega) = |H(\omega)| e^{j\theta(\omega)}$$

$$H(\omega) = \begin{cases} 1 \cdot e^{-j\omega t_0} & \text{for } \omega_c \le |\omega| \le \infty \\ 0 & \text{otherwise} \end{cases}$$

$$H_{LP}(\omega) + H_{HP}(\omega) = H_{AP}(\omega)$$

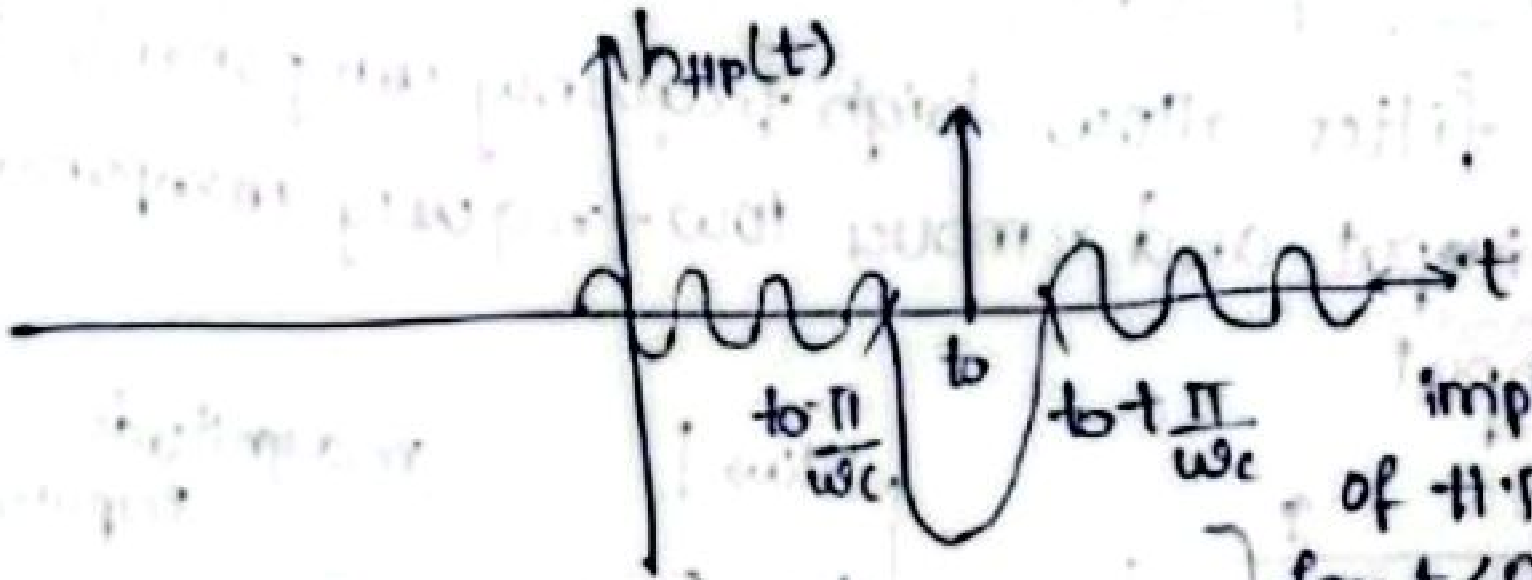
$$H_{HP}(\omega) = H_{AP}(\omega) - H_{LP}(\omega)$$

$$IFT[H_{HP}(\omega)] = IFT[H_{AP}(\omega)] - IFT[H_{LP}(\omega)]$$

$$h_{HP}(t) = h_{AP}(t) - h_{LP}(t)$$

$$\boxed{h_{HP}(t) = \delta(t - t_0) - \frac{\omega_c}{\pi} Sa[\omega_c(t - t_0)]}$$

Impluse response of High pass filter.

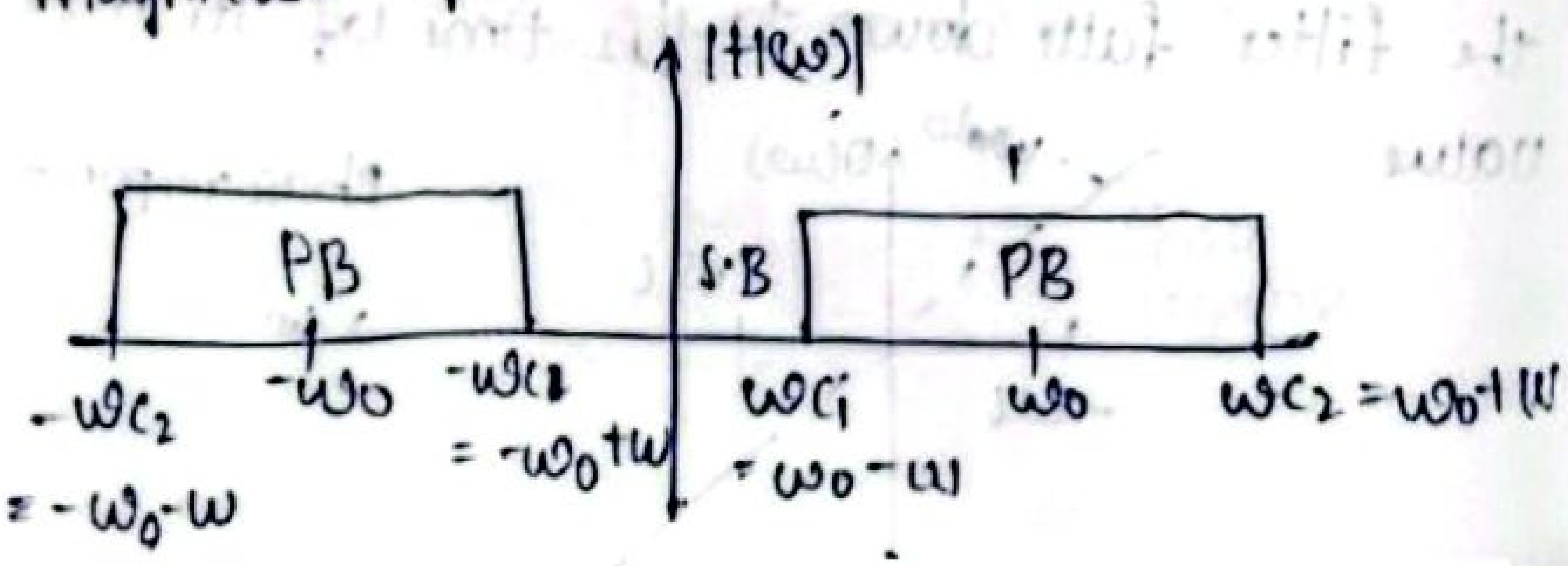


impluse response of H.P filter t_0 for $t<0$: it is non causal system and not physically reasiable.

* Ideal Band pass filter:

Band pass filter allows the Band of frequency of input of the input signal and removes all other frequency of input signal

magnitude response



where ω_0 = centre frequency

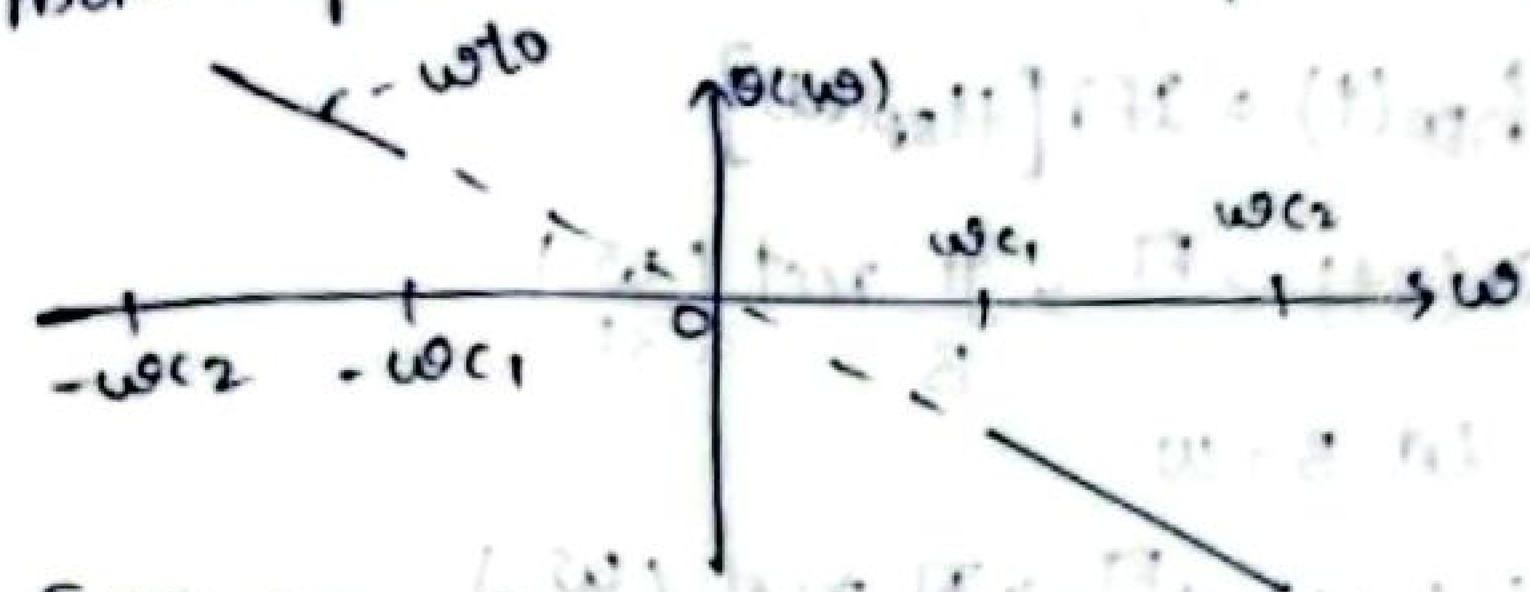
ω_{c_1} = lower cutoff frequency

ω_{c_2} = upper cutoff frequency

Band width of this signal $= \omega_{c_2} - \omega_{c_1}$

$= 2W$

Phase response



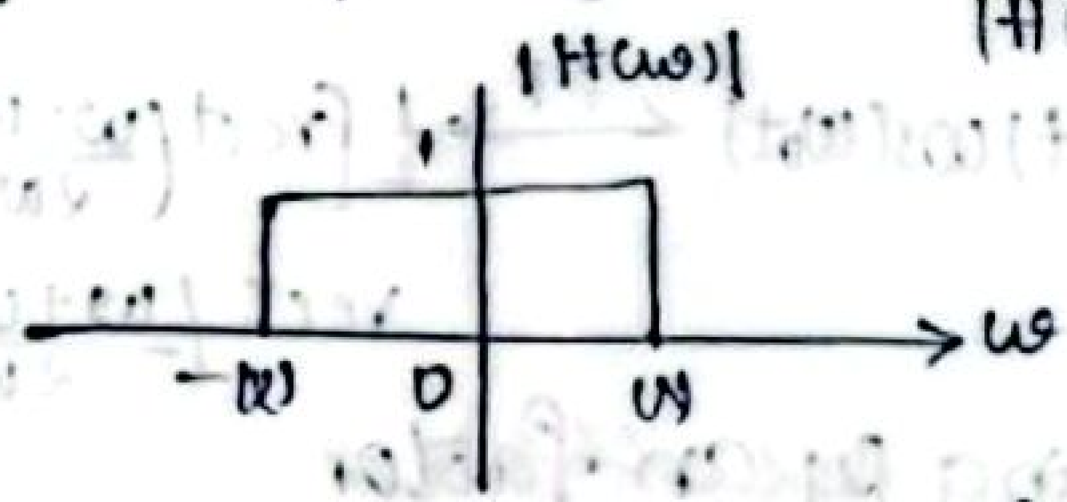
Frequency Response

$$H(\omega) = |H(\omega)| e^{j\theta(\omega)}$$

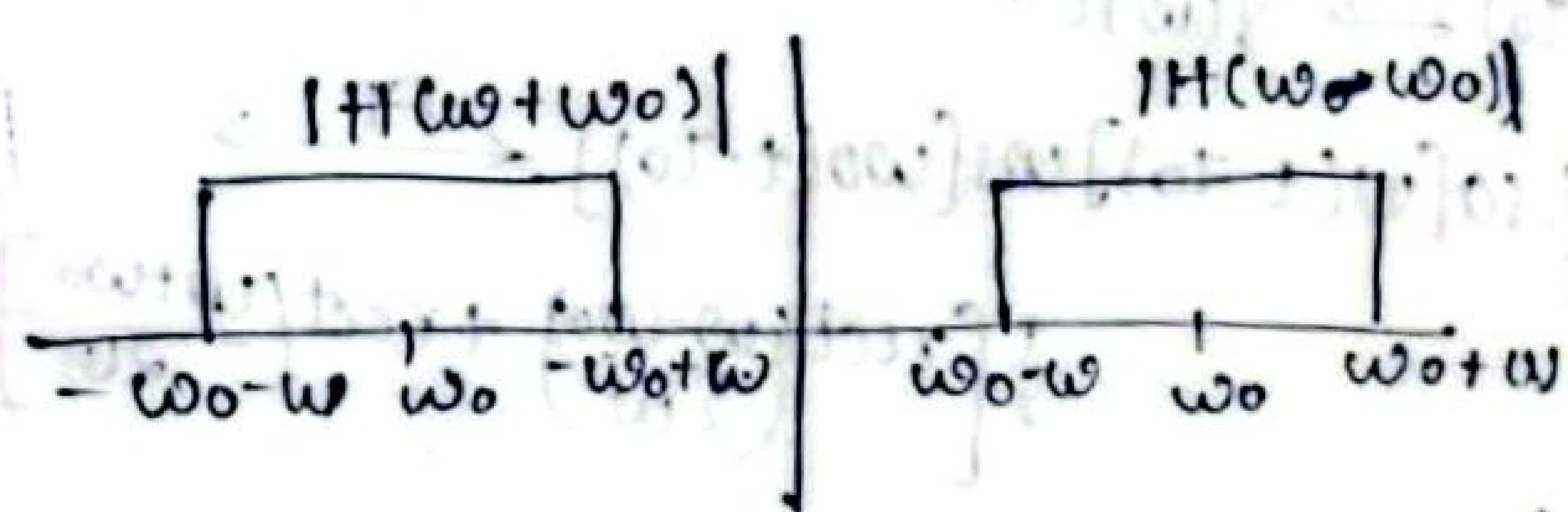
$$H(\omega) = \begin{cases} 1 \cdot e^{-j\omega t_0} & \text{for } \omega_{c_1} \le |\omega| \le \omega_{c_2} \\ 0 & \text{otherwise} \end{cases}$$

Impluse Response.

$$|H(\omega)| = rect\left(\frac{\omega}{2W}\right)$$



By right shift and left shift



$$|H_{BP}(\omega)| = |H(\omega - \omega_0)| + |H(\omega + \omega_0)|$$

$$|H_{BP}(\omega)| = rect\left(\frac{\omega - \omega_0}{2W}\right) + rect\left(\frac{\omega + \omega_0}{2W}\right)$$

frequency response

$H_{BP}(\omega) = |H_{BP}(\omega)| e^{-j\omega t_0}$

$H_{BP}(\omega) = \left[rect\left(\frac{\omega-\omega_0}{2W}\right) + rect\left(\frac{\omega+\omega_0}{2W}\right)\right] e^{-j\omega t_0}$

Impluse response

$h_{BP}(t) = IFT[H_{BP}(\omega)]$

we know, $Sa(Bt) \xleftrightarrow{FT} \frac{\pi}{B} rect\left(\frac{\omega}{2B}\right)$

let $B = W$

$Sa(Wt) \xleftrightarrow{FT} \frac{\pi}{W} rect\left(\frac{\omega}{2W}\right)$

Amplitude scaling by a factor $\frac{W}{\pi}$

$x(t) = \frac{W}{\pi} Sa(Wt) \xleftrightarrow{FT} X(\omega) = rect\left(\frac{\omega}{2W}\right)$

Apply modulation theorem

$x(t)\cos(\omega_0 t) \xleftrightarrow{FT} \frac{1}{2}[X(\omega-\omega_0) + X(\omega+\omega_0)]$

$y(t)\ \frac{W}{\pi} Sa(Wt)\cos(\omega_0 t) \xleftrightarrow{FT} \frac{1}{2}\left[rect\left(\frac{\omega-\omega_0}{2W}\right) + rect\left(\frac{\omega+\omega_0}{2W}\right)\right] = Y(\omega)$

time shifting ~~by~~ factor

$y(t-t_0) \xleftrightarrow{FT} Y(\omega)e^{-j\omega t_0}$

$\frac{W}{\pi} Sa[W(t-t_0)]\cos[\omega_0(t-t_0)] \xleftrightarrow{FT}$

$\frac{1}{2}\left[rect\left(\frac{\omega-\omega_0}{2W}\right) + rect\left(\frac{\omega+\omega_0}{2W}\right)\right]e^{-j\omega t_0}$

Amplitude scaling

$2\frac{W}{\pi} Sa[W(t-t_0)]\cos[\omega_0(t-t_0)] \xleftrightarrow{FT}$

$H_{BP}(\omega) = \left[rect\left(\frac{\omega-\omega_0}{2W}\right) + rect\left(\frac{\omega+\omega_0}{2W}\right)\right]e^{-j\omega t_0}$

$$\boxed{\therefore h_{BP}(t) = \frac{2W}{\pi} Sa[W(t-t_0)]\cos[\omega_0(t-t_0)]}$$

Impluse response of IBPF

Laplace Transforms:

* FT can be used to find the freq info of a signal/system
* LT is a complex F.T & can be used to find the pole-zero response of the system

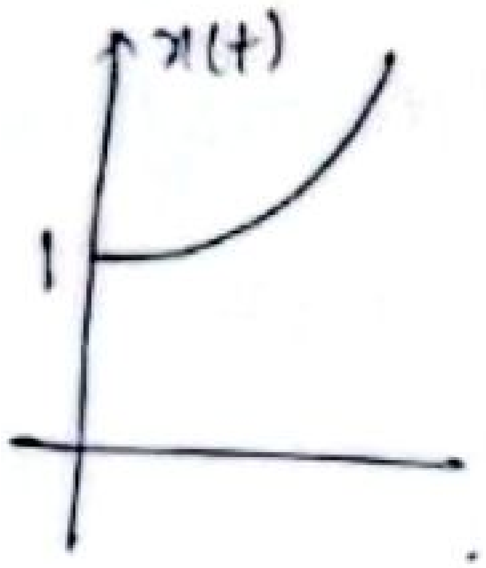
$$x(t) \xleftrightarrow{FT} X(\omega)$$

$$X(\omega) = FT[x(t)] = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt$$

$$x(t) = IFT[X(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} dt$$

* condition for existence of FT => $\int_{-\infty}^{\infty} |x(t)| \cdot dt < \infty$

Ex: $x(t) = e^{2t} u(t)$



$$\int_{-\infty}^{\infty} e^{2t} u(t) dt = \int_{0}^{\infty} e^{2t} dt = \left(\frac{e^{2t}}{2}\right)_0^{\infty}$$

$$= \frac{1}{2}(e^{\infty} - e^{0}) = \infty$$

∴ FT doesn't exist

(∵ It is not absolutely integrable)

$$x(t) \xleftrightarrow{LT} X(s)$$

$X(s) = LT[x(t)]$

$x(t) = ILT[X(s)]$

's' is complex freq variable

$s = a + j\omega$

$$X(s) = LT[X(t)] = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} dt$$

$$= \int_{-\infty}^{\infty} x(t) e^{-(a+j\omega)t} dt$$

$$LT[x(t)] = \int_{-\infty}^{\infty} x(t) e^{-at} \cdot e^{-j\omega t} dt$$

if $\alpha = 0$ => $LT[x(t)] = FT[x(t)]$

condition for existence of LT => $\int_{-\infty}^{\infty} |e^{-\sigma t} \cdot x(t)| dt < \infty$

$x(t) = e^{2t} u(t)$

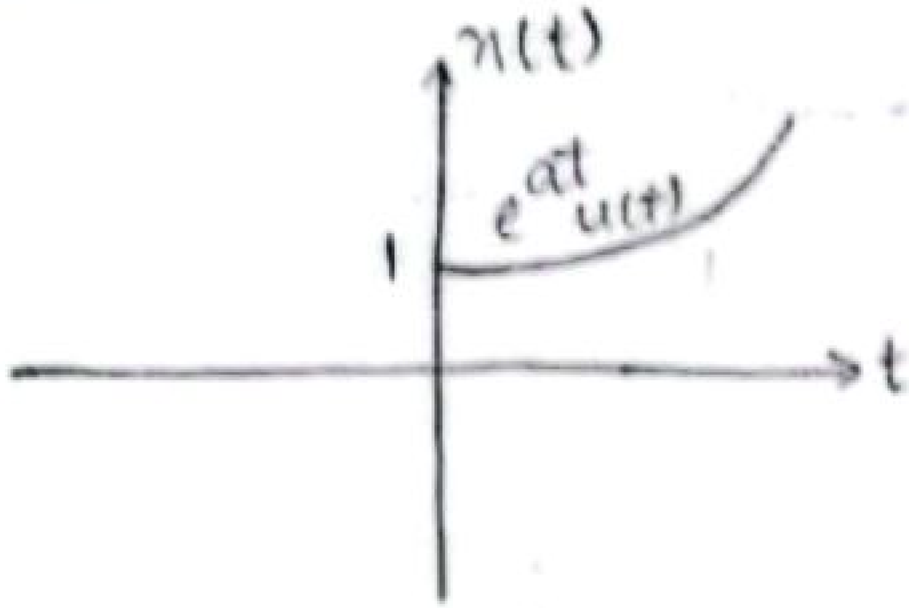
$$\int_{-\infty}^{\infty} |e^{-\sigma t} \cdot e^{2t} u(t)| dt = \int_{0}^{\infty} e^{-(\sigma-2)t} dt$$

$$= \left[\frac{e^{-(\sigma-2)t}}{-(\sigma-2)}\right]_0^{\infty}$$

$$= \frac{-1}{\sigma-2}\left[e^{-(\sigma-2)\infty} - e^{0}\right]$$

$\int_{0}^{\infty} |e^{-\alpha t} \cdot e^{2t} \cdot u(t)$

③ Find the laplace transform of ... where a>0



$$x(s) = LT[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} e^{at} u(t)\, e^{-st}\, dt$$

$$x(s) = \int_{0}^{\infty} e^{-(s-a)t}\, dt$$

$$x(s) = \left. \frac{e^{-(s-a)t}}{-(s-a)} \right|_{0}^{\infty}$$

$$x(s) = \frac{-1}{s-a}\left[e^{-(s-a)\infty} - e^{0}\right]$$

Replace s with $\sigma + j\omega$

$$\therefore x(s) = \frac{-1}{s-a}\left[e^{-(\sigma+j\omega-a)\infty} - 1\right]$$

$$= \frac{1}{s-a}\left[1 - e^{-(\sigma+j\omega-a)\infty}\right]$$

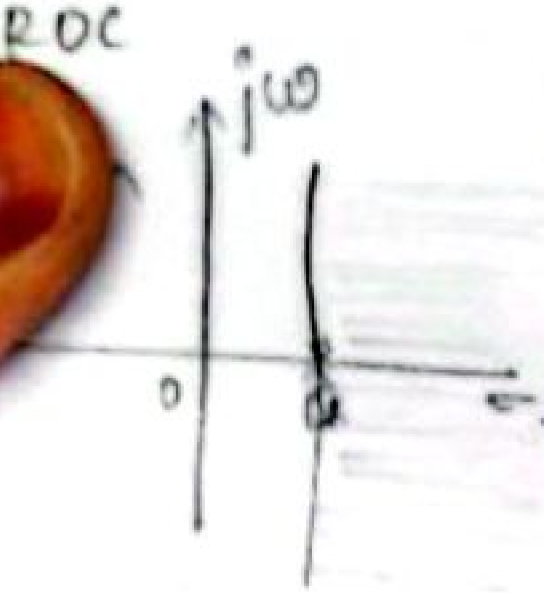
$$= \frac{1}{s-a}\left[1 - \underbrace{e^{-(\sigma-a)\infty}}_{0} \cdot e^{-j\omega\infty}\right]$$

if $\sigma - a > 0$

$\sigma > a$

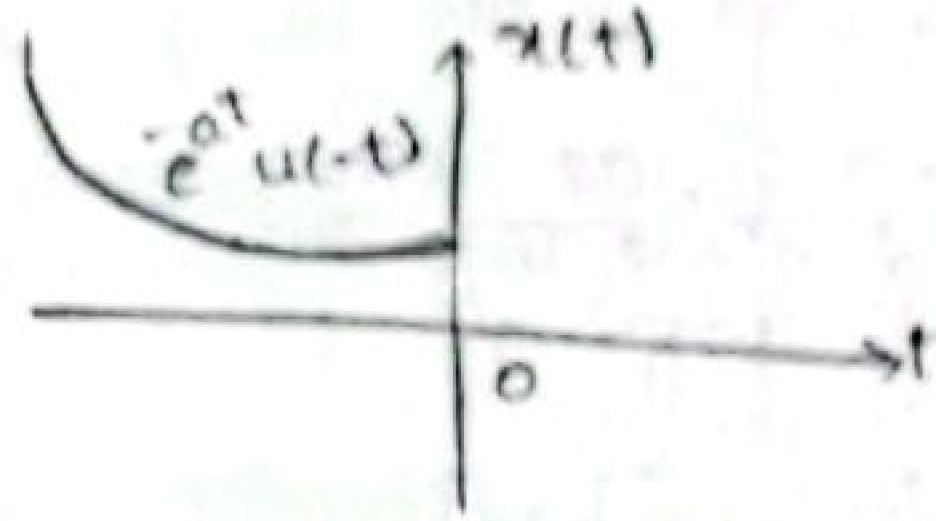
$Re(s) > a$

ROC



$$\boxed{\therefore x(s) = \frac{1}{s-a} \quad ROC;\ \sigma > a}$$

④ Find



$$x(s) = LT[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-at} u(-t)\, dt \cdot e^{-st}$$

$$= \int_{-\infty}^{0} e^{-(s+a)t}\, dt$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_{-\infty}^{0}$$

$$= \frac{-1}{s+a}\left[e^{0} - e^{-(s+a)\infty}\right]$$

$$= \frac{-1}{s+a}\left[1 - e^{-(s+a)\infty}\right]$$

Replace s with $\sigma + j\omega$

$$\therefore x(s) = \frac{-1}{s+a}\left[1 - e^{-(\sigma+j\omega+a)\infty}\right]$$

$$= \frac{-1}{s+a}\left[1 - e^{-(\sigma+a)\infty} \cdot e^{j\omega\infty}\right]$$

$$= \frac{-1}{s+a}\left[1 - 0 \cdot e^{+j\omega\infty}\right]$$

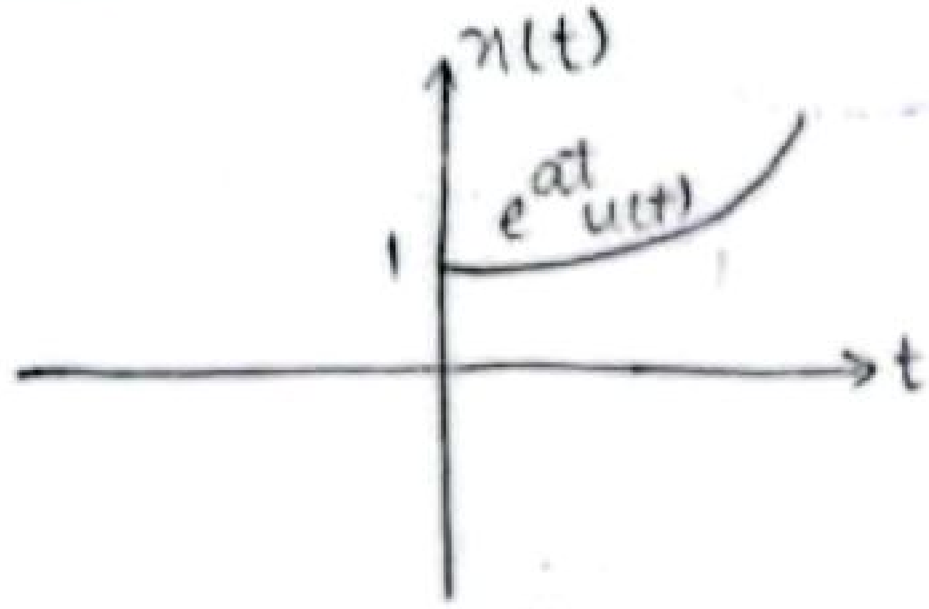
if $\sigma + a < 0$

$\sigma < -a$

$Re(s) < -a$

$$\therefore x(s) = \frac{-1}{s+a} \quad ROC;\ \sigma < -a$$

③ Find the Laplace transform of ... where a>0



$$x(s) = LT[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} e^{at} u(t)\, e^{-st}\, dt$$

$$x(s) = \int_{0}^{\infty} e^{-(s-a)t}\, dt$$

$$x(s) = \left. \frac{e^{-(s-a)t}}{-(s-a)} \right|_{0}^{\infty}$$

$$x(s) = \frac{-1}{s-a}\left[e^{-(s-a)\infty} - e^{0}\right]$$

Replace s with $\sigma + j\omega$

$$\therefore x(s) = \frac{-1}{s-a}\left[e^{-(\sigma+j\omega-a)\infty} - 1\right]$$

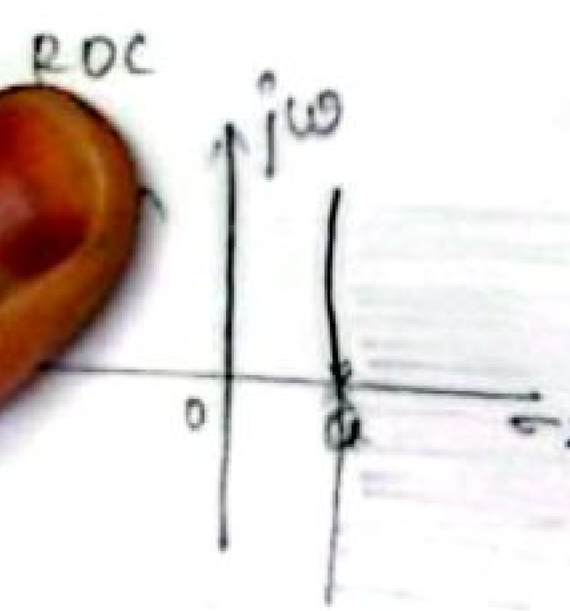
$$= \frac{1}{sa}\left[1 - e^{-(\sigma+j\omega-a)\infty}\right]$$

$$= \frac{1}{s-a}\left[1 - \underbrace{e^{-(\sigma-a)\infty}}_{0} \cdot e^{-j\omega\infty}\right]$$

if $\sigma - a > 0$

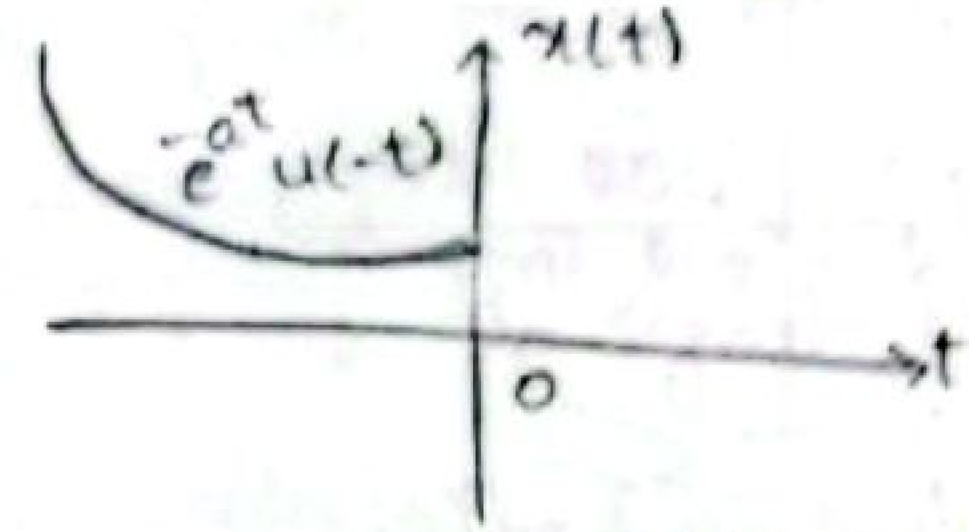
$\sigma > a$

$Re(s) > a$



$$\boxed{\therefore x(s) = \frac{1}{s-a} \quad ROC;\ \sigma > a}$$

④ find



$$x(s) = LT[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-at} u(-t)\, dt \cdot e^{-st}$$

$$= \int_{-\infty}^{0} e^{-(s+a)t}\, dt$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_{-\infty}^{0}$$

$$= \frac{-1}{s+a}\left[e^{0} - e^{-(s+a)\infty}\right]$$

$$= \frac{-1}{s+a}\left[1 - e^{-(s+a)\infty}\right]$$

Replace s with $\sigma + j\omega$

$$\therefore x(s) = \frac{-1}{s+a}\left[1 - e^{-(\sigma+j\omega+a)\infty}\right]$$

$$= \frac{-1}{s+a}\left[1 - e^{-(\sigma+a)\infty} \cdot e^{j\omega\infty}\right]$$

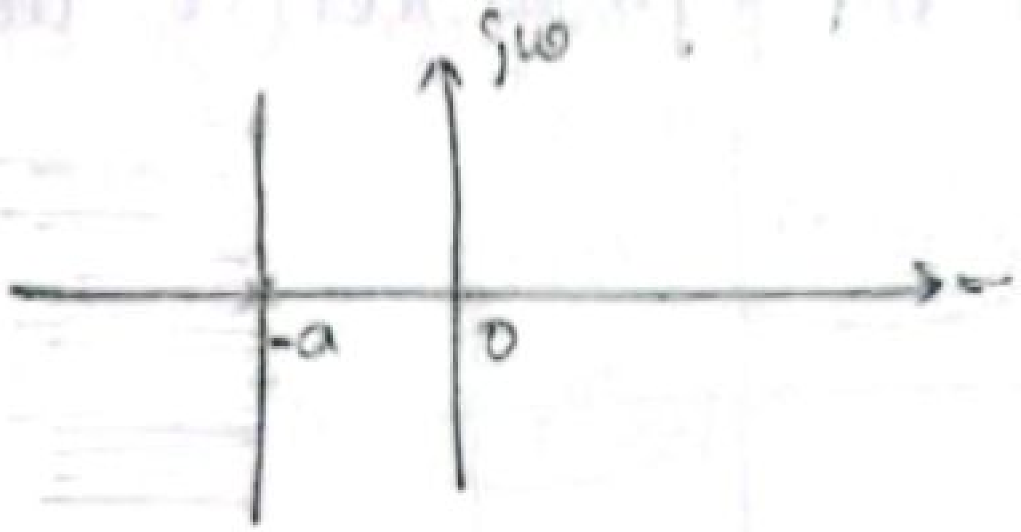
$$= \frac{-1}{s+a}\left[1 - 0 \cdot e^{+j\omega\infty}\right]$$

if $\sigma + a < 0$

$\sigma < -a$

$Re(s) < -a$

$$\therefore x(s) = \frac{-1}{s+a} \quad ROC: \sigma < -a$$



NOTE:

⇒ The pole of the system is denimonator term is zero

① $e^{-at}u(t) \xleftrightarrow{LT} \frac{1}{s+a}$.

pole ⇒ $s+a=0$

$s=-a$

e^{-at} is causal signal

∴ ROC : $\sigma > -a$

② $e^{at}u(-t) \xleftrightarrow{LT} \frac{-1}{s-a}$

pole ⇒ $s-a=0$

$s=a$

e^{at} is anticausal signal

∴ ROC: $\sigma < a$

③ $e^{at}u(t) \xleftrightarrow{LT} \frac{1}{s-a}$,

pole:- $s-a=0$

$s=a$

e^{at} is causal signal

∴ ROC: $\sigma > a$

④ $e^{-at}u(-t) \xleftrightarrow{LT} \frac{-1}{s+a}$

pole:- $s+a=0$

$s=-a$

e^{-at} is anticausal signal

∴ ROC: $\sigma < -a$

*

① for causal signals ROC is $\sigma > \sigma_{max}$

② for anticausal signals ROC is $\sigma < \sigma_{min}$

③ for finite duration signal ROC is entire s-plane

⑤ find the L.T of the signal $x(t) = \delta(t)$

$$x(s) = LT[x(t)] = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$= \int_{-\infty}^{\infty} \delta(t) e^{-st} dt$$

Apply sampling property of an impluse

$$\int_{-\infty}^{\infty} y(t)\delta(t)\, dt = y(t)\Big|_{t=0}$$

$$\therefore x(s) = e^{-st}\Big|_{t=0} = e^0 = 1$$

∴ $x(s) = 1$ ∴ ROC = Entire s-plane

⑥ find the L.T of the signal $x(t) = u(t)$

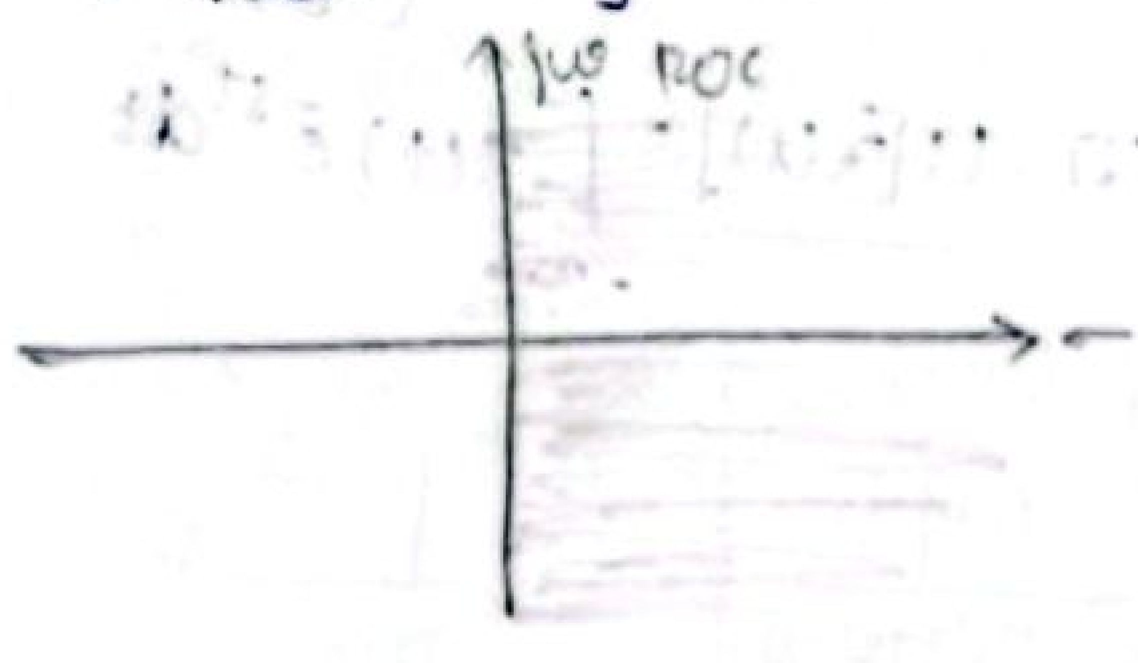
$$x(s) = LT[x(t)] = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

we can consider $e^{-at}u(t)$ (or) $e^{at}u(t)$ where $a=0$

∴ $y(t) = e^{-at}u(t) \xleftrightarrow{LT} y(s) = \frac{1}{s+a}$ ROC: $\sigma > -a$

put $a=0$

∴ $u(t) \xleftrightarrow{LT} \frac{1}{s}$ ROC $\sigma > 0$



find the LT for following

$$x(t) = \begin{cases} e^{at} & \text{for } 0 \le t \le T \\ 0 & \text{otherwise} \end{cases}$$

$$x(s) = LT[x(t)] = \int_{-\infty}^{\infty} x(t)\,e^{-st}\,dt$$

$$= \int_{0}^{T} e^{at} e^{-st}\,dt$$

$$= \int_{0}^{T} e^{(a-s)t}\,dt$$

$$= \left[\frac{e^{(a-s)t}}{(a-s)}\right]_0^T$$

$$= \frac{-1}{-a+s}\left[e^{(a-s)T} - e^{0}\right]$$

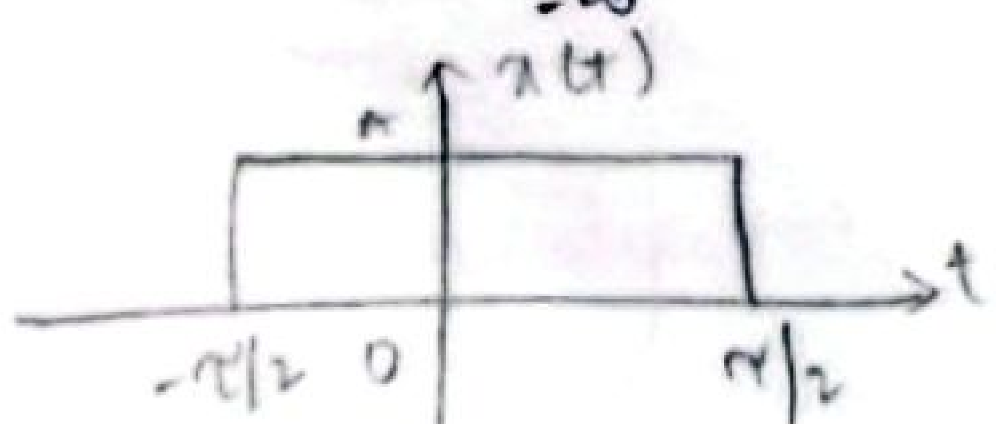
$$x(s) = \frac{1}{s-a}\left[1 - e^{-(s-a)T}\right]$$

∴ ROC: Entire s-plane.

for ③ find the LT of the rectangular pulse

∴ $x(t) = A\,\text{rect}\left(\frac{t}{\tau}\right)$

$$x(s) = LT[x(t)] = \int_{-\infty}^{\infty} x(t)\,e^{-st}\,dt$$



$$= \int_{-\tau/2}^{\tau/2} A\,e^{-st}\,dt$$

$$= A\left[\frac{e^{-st}}{-s}\right]_{-\tau/2}^{\tau/2}$$

$$= \frac{-A}{s}\left[e^{-s(\tau/2)} - e^{s(\tau/2)}\right]$$

$$x(s) = \frac{A}{s}\left[e^{s(\tau/2)} - e^{-s(\tau/2)}\right]$$

∴ ROC is entire s-plane

④ find the ROC for following signal

$$x(t) = e^{-(1-3j)t}u(t)$$

formula: $\int_{-\infty}^{\infty} |e^{-\sigma t} x(t)|\,dt < \infty$

$$\because \int_{-\infty}^{\infty} |e^{-\sigma t} e^{-(1-3j)t} u(t)|\,dt < \infty$$

$$\int_{0}^{\infty} |e^{-\sigma t} e^{-(t)} e^{3jt}|\,dt$$

$$\int_{0}^{\infty} |e^{-(\sigma+1)t}|\,\underbrace{|e^{3jt}|}_{1}\,dt$$

$$\because \int_{0}^{\infty} e^{-(\sigma+1)t}\,dt = \frac{e^{-(\sigma+1)}}{-(\sigma+1)}\Bigg|_0^{\infty}$$

$$= \frac{-1}{\sigma+1}\left[e^{-(\sigma+1)\infty} - e^{0}\right]$$

$\sigma > -1$

$\therefore$ ROC: $\sigma > -1$

(10) find the L.T of the signal $e^{-j\omega_0 t}u(t)$

we know that $e^{-at}u(t) \xleftrightarrow{LT} \frac{1}{s+a}$ ROC: $\sigma > -a$

given $x(t) = e^{-j\omega_0 t}u(t)$

$\therefore e^{-j\omega_0 t}u(t) \xleftrightarrow{LT} \frac{1}{s+j\omega_0}$

$\therefore$ pole: $s + j\omega_0 = 0$

$s = -j\omega_0$

Real part of the pole is zero

$\therefore$ ROC: $\sigma > 0$

(11) $x(t) = e^{j\omega_0 t}u(t)$

$e^{at}u(t) \xleftrightarrow{LT} \frac{1}{s-a}$ ROC: $\sigma > a$

$\therefore e^{j\omega_0 t}u(t) \xleftrightarrow{LT} \frac{1}{s-j\omega_0}$ ROC: $\sigma > 0$

Properties of Laplace transform

(1) Linearity property:

If $x_1(t) \xleftrightarrow{LT} X_1(s)$ ROC: R_1

$x_2(t) \xleftrightarrow{LT} X_2(s)$ ROC: R_2

Then

$\alpha x_1(t) + \beta(x_2(t)) \xleftrightarrow{LT} \alpha X_1(s) + \beta X_2(s)$

ROC: $R_1 \cap R_2$

Proof:

$$L[\alpha x_1(t) + \beta x_2(t)] = \int_{-\infty}^{\infty} (\alpha x_1(t) + \beta x_2(t)) e^{-st}\,dt$$

$$= \int_{-\infty}^{\infty} \alpha x_1(t) e^{-st}\,dt + \int_{-\infty}^{\infty} \beta x_2(t) e^{-st}\,dt$$

$$= \alpha \int_{-\infty}^{\infty} x_1(t) e^{-st}\,dt + \beta \int_{-\infty}^{\infty} x_2(t) e^{-st}\,dt$$

$$= \alpha L[x_1(t)] + \beta L[x_2(t)]$$

$\therefore \alpha x_1(t) + \beta x_2(t) = \alpha X_1(s) + \beta X_2(s)$

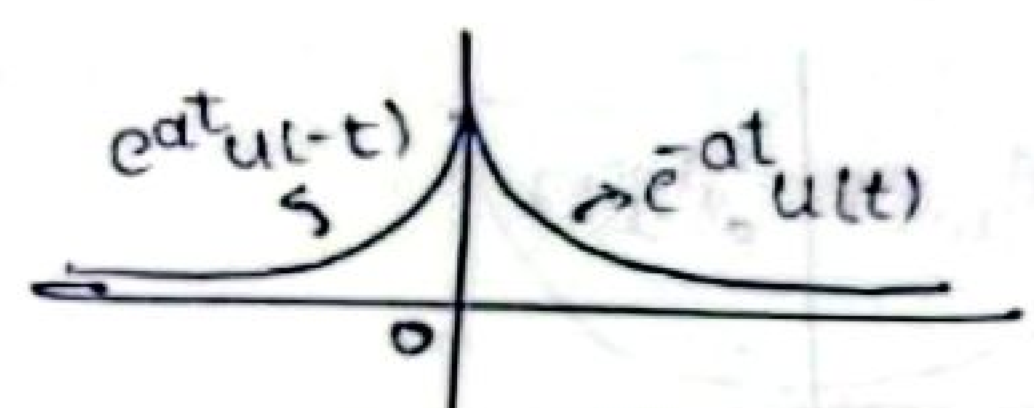
*Ex: Find the L.T of the signal $x(t) = e^{-a|t|}$

(i) $a > 0$, (ii) $a < 0$

(i) $a > 0$

$$x(t) = e^{-a|t|} = \begin{cases} e^{-at} & \text{for } t \geq 0 \\ e^{at} & \text{for } t < 0 \end{cases}$$

$\therefore x(t) = e^{-a|t|} = e^{-at}u(t) + e^{at}u(-t)$

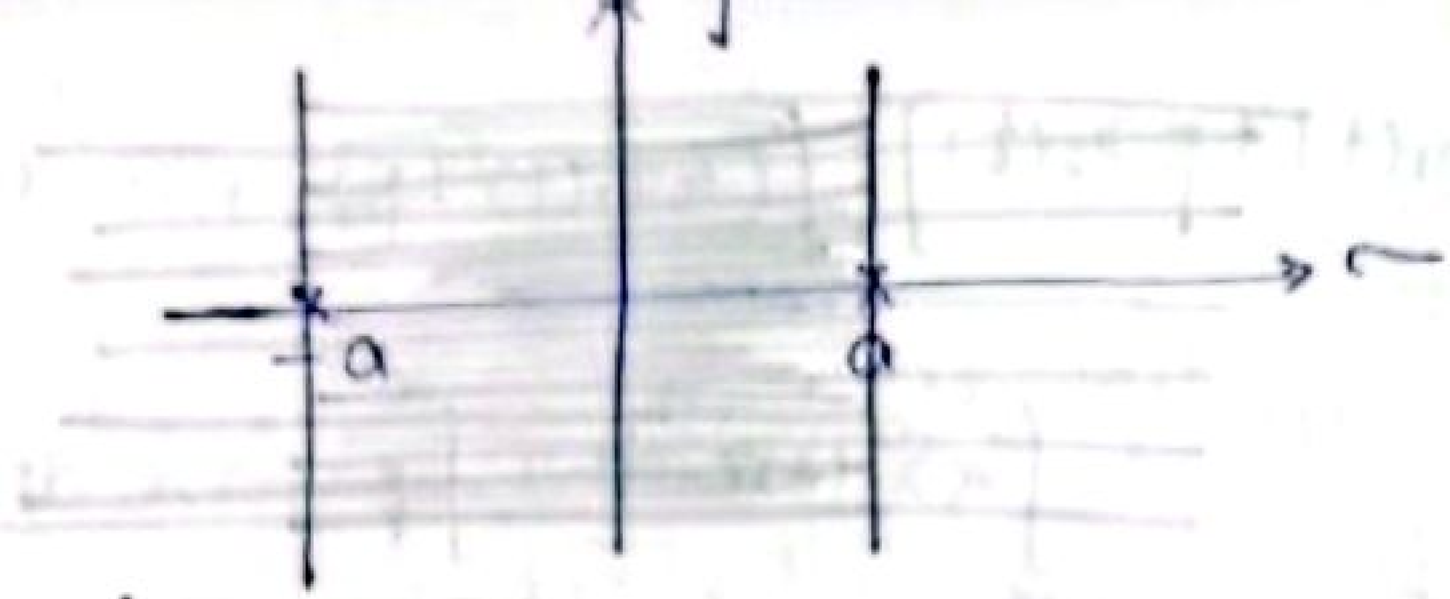


$\therefore$ Apply linearity property

$L[x(t)] = L[e^{-at}u(t)] + L[e^{at}u(-t)]$

$= \frac{1}{s+a} - \frac{1}{s-a}$

ROC: $\sigma > -a$ ROC $\sigma < a$



∴ ROC: $-a < \sigma < a$

and $X(s) = \frac{-2a}{s^2-a^2}$

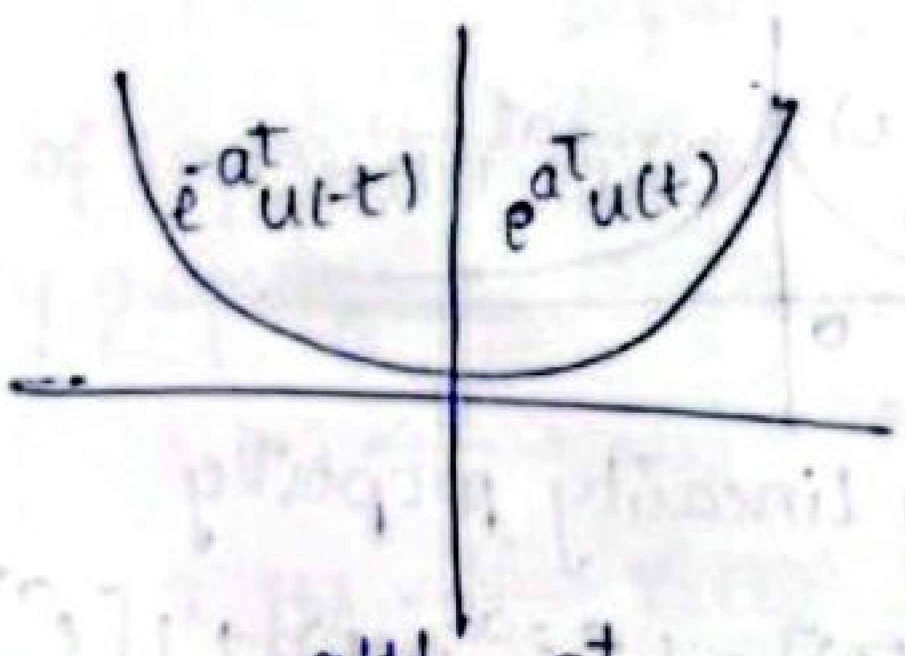
∴ The signal is non causal signal

Note:
Properties of ROC:-

① For two sided signal the ROC lies b/w the two intervals $\sigma_{min} < \sigma < \sigma_{max}$

6 (ii) $a>0$

$$x(t) = e^{-a|t|} = \begin{cases} e^{at} & \text{for } t>0 \\ e^{-at} & \text{for } t<0 \end{cases}$$

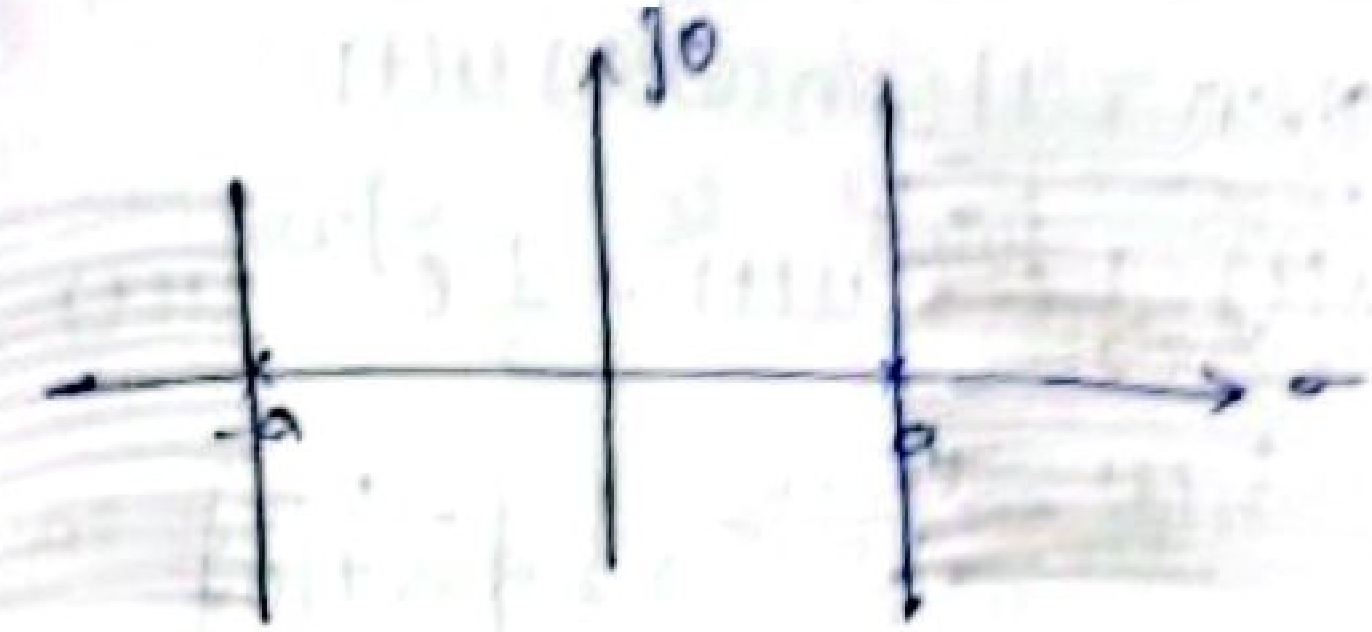


∴ $x(t) = e^{-a|t|} = e^{at}u(t) + e^{-at}u(-t)$

Apply linearity property

$L[x(t)] = L[e^{at}u(t)] + L[e^{-at}u(-t)]$

∴ $X(s) = \frac{1}{s-a} - \frac{1}{s+a}$



∴ common ROC $= \phi$

∴ Laplace transform doesn't exist

⑧ find the L.T of the signal $x(t) = \cos(\omega_0 t)u(t)$

Given $x(t) = \cos(\omega_0 t)u(t)$

$x(t) = \frac{1}{2}e^{j\omega_0 t}u(t) + \frac{1}{2}e^{-j\omega_0 t}u(t)$

$\frac{1}{2}[e^{j\omega_0 t}u(t)] \xleftrightarrow{LT} \frac{1}{2}\left[\frac{1}{s+j\omega_0}\right]$ ROC: $\sigma>0$

$\frac{1}{2}[e^{-j\omega_0 t}u(t)] \xleftrightarrow{LT} \frac{1}{2}\left[\frac{1}{s-j\omega_0}\right]$ ROC: $\sigma>0$

Apply linearity property

$L[x(t)] = \frac{1}{2}L[e^{j\omega_0 t}u(t)] + \frac{1}{2}L[e^{-j\omega_0 t}u(t)]$

∴ $X(s) = \frac{1}{2}\left[\frac{1}{s-j\omega_0}\right] + \frac{1}{2}\left[\frac{1}{s+j\omega_0}\right]$

ROC: $\sigma>0$

$X(s) = \frac{s}{s^2+\omega_0^2}$

$\boxed{\cos(\omega_0 t)u(t) \xleftrightarrow{LT} \frac{s}{s^2+\omega_0^2}}$ ROC: $\sigma>0$

Q find L.T of $x(t) = \sin(\omega_0 t)\ldots$

given $x(t) = \sin(\omega_0 t)\, u(t)$

$$x(t) = \frac{1}{2} e^{j\omega_0 t} u(t) - \frac{1}{2} e^{-j\omega_0 t} u(t)$$

$$\because \frac{1}{2} e^{j\omega_0 t} u(t) \xleftrightarrow{LT} \frac{1}{2}\left[\frac{1}{s+j\omega_0}\right] \quad ROC: \sigma>0$$

$$\frac{1}{2} e^{-j\omega_0 t} u(t) \xleftrightarrow{LT} \frac{1}{2}\left[\frac{1}{s-j\omega_0}\right] \quad ROC: \sigma>0$$

Apply linearity property

$$L[x(t)] = \frac{1}{2j} L\left[e^{j\omega_0 t} u(t)\right] - \frac{1}{2j} L\left[e^{-j\omega_0 t} u(t)\right]$$

$$= \frac{1}{2j}\left[\frac{1}{s+j\omega_0}\right] - \frac{1}{2j}\left[\frac{1}{s-j\omega_0}\right]$$

$\therefore$ ROC : $\sigma>0$

$$\boxed{\because \sin(\omega_0 t) u(t) \xleftrightarrow{LT} \frac{\omega_0}{s^2+\omega_0^2}} \quad ROC: \sigma>0$$

② Time scaling property

If $x(t) \xleftrightarrow{LT} X(s)$

then $x(at) \xleftrightarrow{LT} \frac{1}{|a|} X\left(\frac{s}{a}\right)$ ROC: aR

Proof:

Case①: $a>0$

$-\infty$

let $at = t_1 \Rightarrow t = \frac{1}{a} t_1$

$dt = \frac{1}{a} dt_1$

$$= \int_{-\infty}^{\infty} x(t_1) e^{-s\left(\frac{t_1}{a}\right)} \frac{1}{a} dt_1$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(t_1) e^{-\left(\frac{s}{a}\right)t_1} dt_1$$

$t_1 \to t$

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(t) e^{-\left(\frac{s}{a}\right)t} dt$$

$$\therefore L[x(at)] = \frac{1}{a} X\left(\frac{s}{a}\right)$$

case②:

$a<0$

$L[$

if $x(t) \xleftrightarrow{LT} X(s)$ ROC: R

then $x(-t) \xleftrightarrow{LT} X(-s)$ ROC: −R

Proof:

Time scaling property

$x(at) \xleftrightarrow{LT} \frac{1}{|a|} X\left(\frac{s}{a}\right)$ ROC: aR

Put $a = -1$

$x(-t) \xleftrightarrow{LT} \frac{1}{|-1|} X\left(\frac{s}{-1}\right)$ ROC: −R

$\therefore x(-t) \xleftrightarrow{LT} X(-s)$

Example:

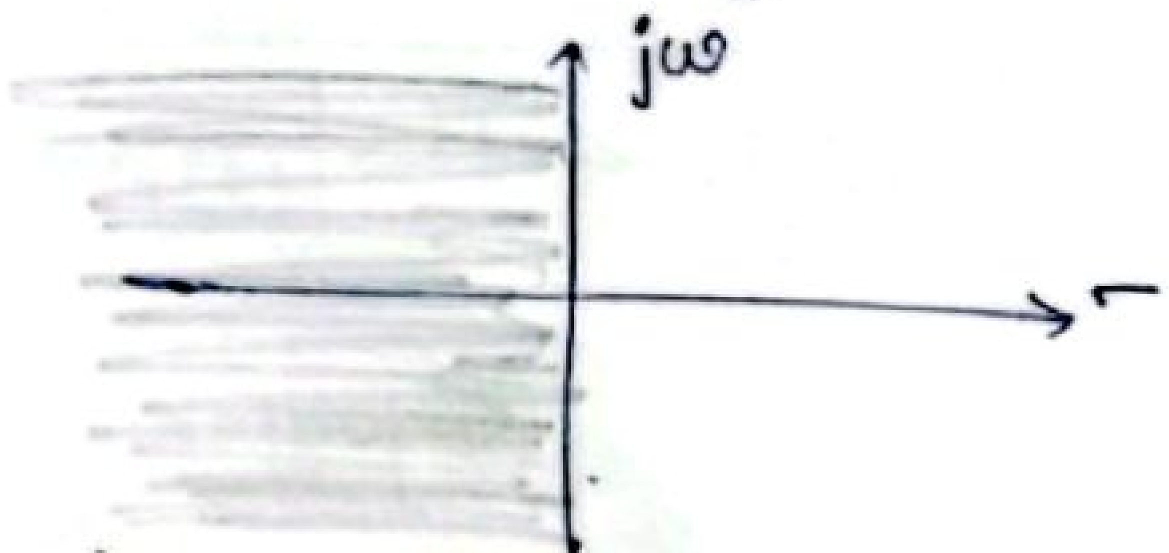
find the LT of the signal $x(t) = u(-t)$

$y(t) = u(t) \xleftrightarrow{LT} \frac{1}{s} = Y(s)$ ROC: $\sigma > 0$

Apply time reversal property

$y(-t) \xleftrightarrow{LT} Y(-s)$

$u(-t) \xleftrightarrow{LT} \frac{-1}{s}$ ROC: $\sigma < 0$



(4) Time shifting property:

If $x(t) \xleftrightarrow{LT} X(s)$ ROC: R

Then $x(t-t_0) \xleftrightarrow{LT} e^{-st_0} X(s)$ ROC: R

Proof:

$$L[x(t-t_0)] = \int_{-\infty}^{\infty} x(t-t_0) e^{-st} dt$$

let $t - t_0 = t_1$

$t = t_1 + t_0$

$dt = dt_1$

$$\therefore \int_{-\infty}^{\infty} x(t_1) e^{-s(t_1+t_0)} dt_1$$

$$\int_{-\infty}^{\infty} x(t_1) e^{-st_1} \cdot e^{-st_0} dt_1$$

$$= e^{-st_0} \int_{-\infty}^{\infty} x(t_1) e^{-st_1} dt_1$$

Replace t_1 with t

$$= e^{-st_0} \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$= e^{-st_0} X(s)$$

$\therefore x(t-t_0) \xleftrightarrow{LT} e^{-st_0} X(s)$ ROC: R

similarly

$x(t+t_0) \xleftrightarrow{LT} e^{st_0} X(s)$ ROC: R

Example:

(1) $x(t) = \delta(t-1)$

$\delta(t) \xleftrightarrow{LT} 1$ ROC: Entire s-plane

$$\therefore L[\delta(t-1)] = \int_{-\infty}^{\infty} \delta(t-1) e^{-st} dt$$

$t - 1 = t_1$

$t = t_1 + 1$

$dt = dt_1$

$$\therefore \int_{-\infty}^{\infty} \delta(t_1) e$$

$$= \int_{-\infty}^{\infty} \delta(t_1) e^{-st_1} e^{-s} dt$$

$$= e^{-s} \int_{-\infty}^{\infty} \delta(t) e^{-st} dt$$

$$= e^{-s}$$

$\therefore \delta(t-1) \xleftrightarrow{LT} e^{-s}$ ROC: Entire S-Plane

② $x(t) = u(t-2)$

$y(t) = u(t) \xleftrightarrow{LT} Y(s) = \frac{1}{s}$ ROC: $\sigma > 0$

Time shifting

$y(t-t_0) \xleftrightarrow{LT} Y(s) e^{-st_0}$

$u(t-2) \xleftrightarrow{LT} \frac{e^{-2s}}{s}$ ROC: $\sigma > 0$

ⓑ shifting in s-Domain property

If $x(t) \xleftrightarrow{LT} X(s)$ ROC: R

Then $e^{-s_0 t} x(t) \xleftrightarrow{LT} X(s+s_0)$ ROC: $R + Re\{s_0\}$

Proof:

$$L[e^{-s_0 t} x(t)] = \int_{-\infty}^{\infty} e^{-s_0 t} x(t) \cdot e^{-st} dt$$

$$= \int_{-\infty}^{\infty} x(t) e^{-(s+s_0)t} dt$$

$$= X(s+s_0)$$

$$\therefore e^{-s_0 t} x(t) \xleftrightarrow{LT} X(s+s_0)$$

$$e^{s_0 t} x(t) \xleftrightarrow{LT} X(s-s_0)$$

$$e^{-at} x(t) \xleftrightarrow{LT} X(s+a)$$

Example:

① $x(t) = e^{-at} \cos(\omega_0 t) u(t)$

$y(t) = \cos(\omega_0 t) u(t) \xleftrightarrow{LT} Y(s) = \frac{s}{s^2+\omega_0^2}$ ROC: $\sigma > 0$

Apply shifting in s-domain property

i.e $e^{-at} y(t) \xleftrightarrow{LT} Y(s+a)$

$$\boxed{e^{-at} \cos(\omega_0 t) u(t) \xleftrightarrow{LT} \frac{s+a}{(s+a)^2+\omega_0^2}}$$

ROC: $\sigma > 0-a$

$\sigma > -a$

Poles $= (s+a)^2 + \omega_0^2 = 0$

$(s+a)^2 = -\omega_0^2$

$s+a = \pm j\omega_0$

$s = -a \pm j\omega_0$

$\therefore$ Real part $= -a$

② $x(t) = e^{-at} \sin(\omega_0 t) u(t)$

$y(t) = \sin(\omega_0 t) u(t) \xleftrightarrow{LT} Y(s) = \frac{\omega}{s^2+\omega_0^2}$ ROC: $\sigma > 0$

Apply shifting in s-domain property

i.e, $e^{-a(t)} y(t) \xleftrightarrow{LT} Y(s+a)$

$\therefore e^{-at} \sin(\omega_0 t) u(t) \xleftrightarrow{LT} \frac{\omega_0}{(s+a)^2+\omega_0^2}$ ROC: $\sigma > 0-a$

$\sigma > -a$

If $x(t) \xleftrightarrow{LT} X(s)$ ROC: R

then $\frac{dx(t)}{dt} \xleftrightarrow{LT} sX(s)$, ROC $\supset$ R

Proof:

ILT

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\,e^{st}\,ds$$ ← def of ILT

differentiate wrt t on both sides

$$\frac{dx(t)}{dt} = \frac{1}{2\pi j}\frac{d}{dt}\left[\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,ds\right]$$

interchange the orders of diff

$$\therefore \frac{dx(t)}{dt} = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\frac{d}{dt}[e^{st}]\,ds$$

$$\frac{dx(t)}{dt} = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}s\cdot ds$$

$$= \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} s\cdot X(s)\,e^{st}\,ds$$

$$\therefore \frac{dx(t)}{dt} = ILT[sX(s)]$$

$$L\left[\frac{dx(t)}{dt}\right] = sX(s)$$

$$\boxed{\frac{dx(t)}{dt} \xleftrightarrow{LT} sX(s)}$$

$$\frac{d^n x(t)}{dt^n} \xleftrightarrow{LT} s^n X(s)$$

(8) solve the following differentiate eqn

$$\frac{d^2y(t)}{dt^2} + y(t) = \delta(t)$$

Given $\frac{d^2y(t)}{dt^2} + y(t) = \delta(t)$

Apply LT on Both sides

$$L\left[\frac{d^2y(t)}{d}\right] + L[y(t)] = L[\delta(t)]$$

$$s^2Y(s) + Y(s) = 1$$

$$Y(s)[s^2+1] = 1$$

$$Y(s) = \frac{1}{s^2+1}$$

we know that

$$\sin(\omega_0 t)u(t) \xleftrightarrow{LT} \frac{\omega_0}{s^2+\omega_0^2}$$

$$y(t) = \sin(t)u(t) \xrightarrow{LT} \frac{1}{s^2+1} = Y(s)$$

$$\therefore y(t) = \sin(t)u(t)$$

(9) $\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2(y(t)) = \delta(t)$

Apply LT on Both side

$$L\left[\frac{d^2y(t)}{dt^2}\right] + 3\cdot L\left[\frac{dy(t)}{dt}\right] + 2\cdot L[y(t)] = L[\delta(t)]$$

$$s^2Y(s) + 3sY(s) + 2Y(s) = 1$$

$$Y(s) = \frac{1}{s^2+3s+2}$$

$$Y(s) = \frac{1}{(s+1)(s+2)}$$

Apply partial fractions

$$Y(s) = \frac{1}{(s+1)(s+2)} = \frac{C_1}{s+1} + \frac{C_2}{s+2}$$

$$\therefore Y(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

Apply ILT

$$y(t) = e^{-t}u(t) - e^{-2t}u(t)$$

⑦ differentiation in s-domain property

If $x(t) \xleftrightarrow{LT} X(s)$ ROC: R

Then $-t\,x(t) \xleftrightarrow{LT} \frac{dX(s)}{ds}$ ROC $\supset$ R

__Proof:__

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

integrat w.r.t s on both sides

$$\frac{dX(s)}{ds} = \frac{d}{ds}\int_{-\infty}^{\infty} x(t)\,e^{-st}dt$$

$$\frac{dX(s)}{ds} = \int_{-\infty}^{\infty} x(t)\,\frac{d}{ds}e^{-st}dt$$

$$= \int_{-\infty}^{\infty} x(t)\cdot(-t)e^{-st}dt$$

ds

$$\boxed{-t\,x(t) \xleftrightarrow{LT} \frac{dX(s)}{ds}}$$

In general

$$\boxed{t^n x(t) \xleftrightarrow{LT} (-1)^n\frac{d^nX(s)}{ds^n}}$$

Q $x(t) = t\cdot e^{-at}u(t)$

we know that

$$y(t) = e^{-at}u(t) \xleftrightarrow{LT} \frac{1}{s+a} = Y(s)$$

Apply diff in s-domain property

$$-t\,y(t) \xleftrightarrow{LT} \frac{dY(s)}{ds}$$

$$-t\cdot e^{-at}u(t) \xleftrightarrow{LT} \frac{-1}{(s+a)^2}$$

Amplitud scaling by an factor -1

$$\boxed{\therefore t\,e^{-at}u(t) \xleftrightarrow{LT} \frac{1}{(s+a)^2}}$$ $\therefore$ ROC: $\sigma > -a$

Q Find the L.T of Ramp signal $[x(t) = t\,u(t)]$

we know that

$$t\,e^{-at}u(t) \xleftrightarrow{LT} \frac{1}{(s+a)^2}$$

Put $a=0$

$$\therefore t\,u(t) \longleftrightarrow \frac{1}{s^2}$$ $\therefore$ ROC: $\sigma > 0$

Q $x(t) = t^2u(t)$

we know that

$$y(t) = t\cdot u(t) \xleftrightarrow{LT} Y(s) = \frac{1}{s^2}$$

apply diff in s-domain

$$-t\,y(t) \xleftrightarrow{LT} \frac{dY(s)}{ds}$$

$$-t^2 u(t) \xleftrightarrow{LT} \frac{-2}{s^3}$$

$$t^2 u(t) \xleftrightarrow{LT} \frac{2}{s^3} = \frac{2!}{s^3}$$

simillary

$$t^3 u(t) \xleftrightarrow{LT} \frac{6}{s^4} = \frac{3!}{s^4}$$

∴ In general

$$\boxed{\frac{t^n}{n!} u(t) \xleftrightarrow{LT} \frac{1}{s^{n+1}}}$$

NOTE:

$$\frac{t^n}{n!} \cdot e^{-at} u(t) \xleftrightarrow{LT} \frac{1}{(s+a)^{n+1}}$$

By using shifting in s-domain property

(8) convolution theorem

If $x_1(t) \xleftrightarrow{LT} X_1(s)$; ROC: R_1

$x_2(t) \xleftrightarrow{LT} X_2(s)$; ROC: R_2

then $x_1(t) * x_2(t) \xleftrightarrow{LT} X_1(s) X_2(s)$; ROC: $R_1 \cap R_2$

Proof: $L[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} [x_1(t) * x_2(t)] e^{-st} dt$

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau) d\tau$$

$$L[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau) d\tau\right] e^{-st} dt$$

$$= \int_{-\infty}^{\infty} x_1(\tau) \left[\int_{-\infty}^{\infty} x_2(t-\tau) e^{-st} dt\right] d\tau$$

$$= \int_{-\infty}^{\infty} x_1(\tau) e^{-s\tau} X_2(s) d\tau$$

$$= X_2(s) \int_{-\infty}^{\infty} x_1(\tau) e^{-s\tau} d\tau$$

$$L[x_1(t) * x_2(t)] = X_2(s) \cdot X_1(s)$$

$$x_1(t) \cdot x_2(t) = L^{-1}[X_1(s) X_2(s)]$$

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty}$$

$$\frac{t^n}{n!} e^{-at} u(t) \longrightarrow \frac{1}{(s+a)^{n+1}}$$

- Apply shifting in S-domain.

Convolution theorem:-

If $x_1(t) \xleftrightarrow{LT} X_1(s)$ ROC: R_1

$x_2(t) \xleftrightarrow{LT} X_2(s)$ ROC: R_2

then $x_1(t) * x_2(t) \xleftrightarrow{LT} X_1(s) X_2(s)$, ROC: $R_1 \cap R_2$

Proof:-

$$L[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} [x_1(t) * x_2(t)] e^{-st} dt$$

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau) d\tau$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau) d\tau \right] e^{-st} dt$$

$$= \int_{-\infty}^{\infty} x_1(\tau) \underbrace{\left(x_2(t-\tau) e^{-st} dt \right)}_{L[x_2(t-\tau)] = e^{-s\tau} X_2(s)} d\tau$$

= Apply time shifting property

$$= \int_{-\infty}^{\infty} x_1(\tau) e^{-s\tau} X_2(s) d\tau$$

$$= X_2(s) \underbrace{\int_{-\infty}^{\infty} x_1(\tau) e^{-s\tau} d\tau}_{L[x_1(t)] = X_1(s)}$$

$$L[x_1(t) * x_2(t)] = X_1(s) X_2(s)$$

$$x_1(t) * x_2(t) = L^{-1}[X_1(s) X_2(s)]$$

$$x_1(t) * x_2(t) = \int x_1(\tau) x_2(t-\tau) d\tau = L^{-1}[X_1(s) X_2(s)]$$

$L[x_1(t) * x_2(t)] = L[x_1(t)] \cdot [x_2(t)]$

$x_1(t) * x_2(t) \xleftrightarrow{LT} X_1(s) \cdot X_2(s)$

Find the convolution b/w the two following signals. $x_1(t) = u(t)$ & $x_2(t) = u(t)$ using convolution ~~using~~ ~~convo~~ theorem.

Sol

$x_1(t) = u(t)$

$x_2(t) = u(t)$

using Ramp signal

$x_1(t) = u(t) \xleftrightarrow{LT} X_1(s) = \frac{1}{s}$ $\sigma > 0$

$x_2(t) = u(t) \xleftrightarrow{LT} X_2(s) = \frac{1}{s}$ $\sigma > 0$

$x_1(t) * x_2(t) = L^{-1}[X_1(s) X_2(s)]$

$= L^{-1}\left[\frac{1}{s^2}\right]$ $\sigma > 0$

$= t \cdot u(t)$

Find the convolution b/w the two following signals. $x_1(t) = e^{-at}u(t)$

$x_2(t) = e^{-bt}u(t)$

Sol Given $x_1(t) = e^{-at}u(t)$

$x_2(t) = e^{-bt}u(t)$.

$x_1(t) = e^{-at}u(t) \xleftrightarrow{LT} X_1(s) = \frac{1}{s+a}$ $\sigma > -a$

$x_2(t) = e^{-bt}u(t) \longleftrightarrow X_2(s) = \frac{1}{s+b}$ $\sigma > -b$

$x_1(t) * x_2(t) = L^{-1}[X_1(s) \cdot X_2(s)]$

$= L^{-1}\left[\left(\frac{1}{s+a}\right)\left(\frac{1}{s+b}\right)\right]$

$\frac{1}{(s+a)(s+b)} = \frac{C_1}{s+a} + \frac{C_2}{s+b}$

$1 = C_1(s+b) + C_2(s+a)$

put $s = -a$

$1 = C_1(b-a) \Rightarrow C_1 = \frac{1}{b-a}$

put $s = -b$

$C_2 = \frac{1}{b-a}$

$X(s) = \frac{1}{b-a}\left[\frac{1}{s+a} - \frac{1}{s+b}\right]$

$x(t) = \left[\frac{e^{-at}u(t) - e^{-bt}u(t)}{b-a}\right]$

3) Multiplication Theorem (Convolution in S-domain):

If $x_1(t) \xleftrightarrow{LT} X_1(s)$ ROC: R_1

$x_2(t) \xleftrightarrow{LT} X_2(s)$ ROC: R_2

$x_1(t) * x_2(t) \xleftrightarrow{LT} \frac{1}{2\pi j}[X_1(s) * X_2(s)]$ ROC: $R_1 \cap R_2$

Proof: ILT

$x_1(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X_1(\lambda)e^{\lambda t}d\lambda$

$L[x_1(t) \cdot x_2(t)] = \int_{-\infty}^{\infty}[x_1(t)\, x_2(t)]e^{-st}dt$

$= \int_{-\infty}^{\infty}\left[\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X_1(\lambda)e^{\lambda t}d\lambda\right]x_2(t)e^{-st}dt$

$= \frac{1}{2\pi j}\int_{-\infty}^{\infty} x_1(\lambda)\left[\int_{-\infty}^{\infty} x_2(t)e^{-(s-\lambda)t}dt\right]d\lambda$

$= \frac{1}{2\pi j}\int_{-\infty}^{\infty} x_1(\lambda)x_2(s-\lambda)d\lambda$

$$= \frac{1}{2\pi j}\left[X_1(s) * X_2(s)\right]$$

$$x_1(t)\,x_2(t) \xleftrightarrow{LT} \frac{1}{2\pi j}\left[X_1(s) * X_2(s)\right]$$

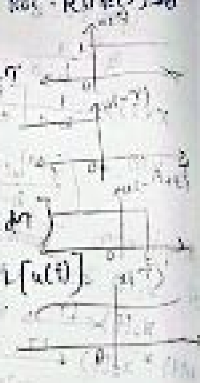
Integration in time domain property:

If $x(t) \xleftrightarrow{LT} X(s)$ ROC: R

then $\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{LT} \frac{X(s)}{s}$ ROC: $R \cap Re(s) > 0$

$$x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau)\,u(t-\tau)\,d\tau$$

$$x(t) * u(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$$



Apply Laplace on both sides

$$L\left[x(t) * u(t)\right] = L\left[\int_{-\infty}^{t} x(\tau)\,d\tau\right]$$

$$L\left[\int_{-\infty}^{t} x(\tau)\,d\tau\right] = L\left[x(t)\right] \cdot L\left[u(t)\right]$$

$$= X(s)\,\frac{1}{s}$$

$$\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{LT} \frac{X(s)}{s}$$

$\int_{0}^{t} y(t)\,dt = 3u(t) - 2y(t)$, find $y(t)$

$$L\left[\int_{0}^{t} y(t)\,dt\right] = 3L\left[u(t)\right] - 2L\left[y(t)\right]$$

$$\frac{Y(s)}{s} = \frac{3}{s} - 2Y(s)$$

$$\frac{Y(s)}{s} + 2Y(s) = \frac{3}{s}$$

$$Y(s)\left[\frac{1}{s} + 2\right] = \frac{3}{s}$$

$$Y(s)\left[\frac{1+2s}{s}\right] = 3/s$$

$$Y(s) = \frac{3}{1+2s} = \frac{3/2}{s+1/2}$$

$$\boxed{y(t) = \frac{3}{2}\,e^{-1/2\,t}\,u(t)}$$

Integration in S-domain property:

If $x(t) \xleftrightarrow{LT} X(s)$ ROC: R

then $\frac{x(t)}{t} \xleftrightarrow{LT} \int_{s}^{\infty} X(s)\,ds$ ROC: R.

Proof:

$$X(s) = \int_{-\infty}^{\infty} x(t)\,e^{-st}\,dt$$

$$\int_{s}^{\infty} X(s)\,ds = \int_{s}^{\infty}\left[\int_{-\infty}^{\infty} x(t)\,e^{-st}\,dt\right]ds$$

$$= \int_{-\infty}^{\infty}\left[x(t)\int_{s}^{\infty} e^{-st}\,ds\right]dt$$

$$\int_{s}^{\infty} e^{-st}\,ds = \left[\frac{e^{-st}}{-t}\right]_{s}^{\infty} = \frac{-1}{t}\left[e^{-\infty t} - e^{-st}\right]$$

$$\int_{s}^{\infty} X(s)\,ds = \int_{-\infty}^{\infty} \frac{x(t)}{t}\,e^{-st}\,dt$$

$$\int_{s}^{\infty} X(s)\,ds = L\left[\frac{x(t)}{t}\right]$$

$$\frac{x(t)}{t} \xleftrightarrow{LT} \int_{s}^{\infty} X(s)\,ds$$